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Elementary Students' Use of Relationships and Physical Models to Understand Order and Equivalence of Rational Numbers

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Elementary Students' Use of Relationships and Physical Models to Understand Order and Equivalence of Rational Numbers

by

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Dissertation

Presented to the Faculty of the Graduate School of

The University of Texas at Austin

in Partial Fulfillment

of the Requirements

for the Degree of

Doctor of Philosophy

The University of Texas at Austin

August 2003

Dedication

For all of my teachers who encouraged my love of learning and inspired me to strive for understanding

Acknowledgements

This dissertation would not have been possible without the teacher and students who participated in this study. The faculty and students at the school welcomed me and let me become a part of their amazingly rich learning environment where teaching children to think and question is highly valued. The children in the study always worked hard and answered my questions thoughtfully. They made collecting data a wonderfully exciting and interesting process. I know I learned as much from observing the teacher as the students did from being in her class. I am honored that through this process, the teacher has become a trusted colleague and friend.

A heartfelt thank you goes to each member in my peer debriefing group: Cindy Schneider, Christy Youker, Candace Figg, and Joanne Williams. They have helped me through every stage in the process of this dissertation. Each of these friends provided support and assistance whenever I asked, even though they all had their own busy lives. Their moral support has been invaluable.

I am thankful for the editing help my friend Daniel Beck provided on several sections of my dissertation. He called me the "passive queen" at one point, but I have learned how to use the active voice based on his feedback. I know that my dissertation is easier to read and understand because of his assistance.

Thank you to each of my committee members for providing support and encouragement throughout this process. Susan Empson, my dissertation chair, deserves a special thank you for her guidance and sharing her expertise especially in the areas of children's thinking and fraction learning. All of her feedback and suggestions in the analysis and the writing of my dissertation have improved the quality of my research and writing.

I want to thank my parents, Ted Wallace and Lupe Haro Wallace, for encouraging me to always do my best. Based on the values they taught me growing up, I believe in the importance of education for all children. I know I am fortunate to be in a career path where I can make a difference in other people's lives.

Finally, I am truly grateful to my husband, Eric Wenrick, who has been supportive and encouraging all the time that I have been in graduate school and working on my dissertation. He has helped by listening to me when I was trying to organize my ideas, giving me feedback on chapters, and being a calming force when my work became overwhelming. He has also provided technical support, especially by developing the database for organizing and analyzing the very complex data. I know that I am blessed to have Eric as my best friend and life partner.

Elementary Students' Use of Relationships and Physical Models to Understand Order and Equivalence of Rational Numbers

Publication No._____________

Melanie Renee Wenrick, Ph.D.
The University of Texas at Austin, 2003

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This study examined the interaction between the use of physical models and children's understanding of fractions as demonstrated through their ability to compare and order fractions. Although physical models are recommended to help children developing an understanding of fraction concepts, there are multiple ideas about how to use the materials in classroom instruction and the results concerning the effectiveness of physical models have been mixed. Post, Behr, and Lesh (1986) suggested students must develop a "quantitative notion of rational number" (p. 40) which was directly connected to their ability to compare and order fractions. Smith (1995) identified four perspectives (Parts, Components, Reference Points, and Transform) to categorize general approaches for solving order and equivalence problems, which provided a framework for this study.

Thirteen students from a multi-level third, fourth, and fifth grade class participated in this study. The teacher's mathematics instruction was organized around problem solving and discussion of solutions. Daily classroom observations were videotaped over a three month period. All thirteen students participated in individual clinical interviews prior to and after the unit; eight students participated in interviews midway through the unit. All interviews were videotaped and summarized.

The analysis of the data identified the relationships students attended to when comparing and ordering fractions. These relationships were grouped into eight perspectives (Limited, Pieces, Part-Whole, Unit Fraction, Within-Fraction, Between-Fraction, Equivalence, and Transform) extending Smith's (1995) work. Many of these perspectives were connected to developing a quantitative notion of fractions and were influenced by the use of physical models. Physical models were used for more than just finding answers. Pre-partitioned area and linear models helped students learn equivalent relationships; however, some students acted as though the pieces were unrelated to the whole or used materials without thinking about relationships. Relationships were extremely important for comparing and ordering fractions in the part-whole perspective. Students who were able to identify, use, and extend relationships had a stronger understanding of fractions and could move between perspectives to solve problems efficiently and effectively.

Table of Contents

List of Tables

List of Figures

Chapter 1: Introduction

The *Principles and Standards for School Mathematics* (National Council of Teachers of Mathematics, 2000) states that, "Representing numbers with various physical materials should be a major part of mathematics instruction in the elementary school grades" (p. 33). Yet research has demonstrated that children do not automatically understand the relationship between concrete models and the underlying mathematical concept (Gravemeijer, 1997; Thompson & Lambdin, 1994), even though these relationships are readily apparent to adults who understand the concept (Behr, Lesh, Post, & Silver, 1983; Gravemeijer, 1997; 2000). For example, adults who understand rational number concepts are more likely than children to realize that paper folding depicts the multiplicative relationships inherent in equivalent fractions. This raises the question: What relationships do children understand when they work with physical models?

USING PHYSICAL MODELS WITH FRACTIONS

The National Council of Teachers of Mathematics (NCTM) Standards (2000) recommends that teachers use physical models in third through fifth grades to develop students' understanding of fractions including learning about relative size. In addition, NCTM advises that students use physical models to do simple computations with fractions. Numerous articles geared towards teachers illustrate specific methods and activities for teaching fraction concepts and computation using models (Bezuk & Bieck, 1993; Caldwell, 1995; Cramer & Bezuk, 1991; Ott, Snook, & Gibson, 1991; Yvonne Pothier & Sawada, 1990; Schultz, 1991; Zeman, 1991). At the same time, Thompson (1994) warns that teaching students a process for using concrete materials does not necessarily lead to students understanding the concepts.

Since teachers frequently use physical models to teach fraction concepts, researchers have examined the use of different types of physical models and their effectiveness. Some studies have identified which classes of physical models for fractions were easier and more difficult for children to understand (Behr, Wachsmuth, & Post, 1988; D'Ambrosio & Mewborn, 1994; Larson, 1980; Pitkethly & Hunting, 1996). Other studies have focused on how physical materials were used in instruction (Behr et al., 1988; Bezuk & Bieck, 1993; Connell & Peck, 1993; Post, Wachsmuth, Lesh, & Behr, 1985). Researchers have also proposed specific steps to help students move from using physical models to using symbolic notation (Behr et al., 1988; Bezuk & Bieck, 1993; Connell & Peck, 1993) and others have examined the connections that students made between models and symbols (Brinker, 1997; D'Ambrosio & Mewborn, 1994; Hope & Owens, 1987). Several studies assessed the effectiveness of instruction based on using multiple physical models (Behr, Wachsmuth, Post, & Lesh, 1984; Cramer, Post, & delMas, 2002; Cramer, Post, Henry, & Jeffers-Ruff, 2000; Post et al., 1985). While these studies have helped classify the difficulties of various physical models and provided evidence of their effectiveness, they have minimally described the interaction between the physical models and children's understanding of fraction concepts.

Instead of focusing only on what physical models to use or how to use them, researchers should ask another broader question: How do physical models influence children's understanding of fractions? As mathematics educators, we need to understand more about the connections between children's thinking about fractions and their use of physical models before we can determine effective ways to use physical models in the elementary mathematics class.

UNDERSTANDING FRACTIONS

Skemp (1978) described two kinds of understanding: relational and instrumental. Instrumental understanding is knowing a procedure and being able to use it, while relational understanding is "knowing both what to do and why" (p. 9). Relational understanding helps students make connections that increases retention, reduces the need to remember procedures or rules for every situation, and allows them to transfer their learning to new situations (Hiebert & Carpenter, 1992; Skemp, 1978). Relational understanding requires identifying and using relationships between multiple pieces of information. This dissertation focused on relational understanding.

Hiebert and Carpenter (1992) presented a framework for relational understanding that is based on both external and internal representations. External representations include concrete materials, oral language, pictures, and symbols, but as Hiebert and Carpenter (1992) point out, these relationships can highlight certain aspects of the mathematical system while obscuring others. Even though internal representations are more difficult to categorize because they are only observed indirectly through external representations, the connections are described as "networks of knowledge" (Hiebert & Carpenter, 1992, p. 67). These networks are described metaphorically as spider webs where relationships are the threads in the web and as hierarchies where relationships between general and specific representations are connected vertically. Hiebert and Carpenter (1992) explained that, “the degree of understanding is determined by the number and strength of the connections” (p. 67). These connections are based on the relationships that students identify and use as they solve problems. Lesh and Doerr (2000) suggested that “conceptual systems seldom function without the support of powerful tools or representational systems, each of which emphasizes or deemphasizes (or ignores or distorts) somewhat different aspects of the underlying conceptual system” (p. 362).

Hiebert and Carpenter (1992) also point out that “Many of those who study mathematics learning agree that understanding involves recognizing relationships between pieces of information” (p. 67). For children to develop an understanding of fractions, they need to make connections between internal and external representations based on their informal knowledge, previous experience, classroom instruction and use of external representations. Understanding various characteristics of rational numbers, including decimals, percents and fractions, will help children develop what Post et al. (1986) call "a quantitative notion of rational number" (p. 40). To develop a quantitative notion, students must identify and use number relationships relevant to rational numbers such as relationships between the numerator and denominator in equivalent fractions. Many of the number relationships that are part of developing a quantitative notion of rational numbers are also important for learning to compare and order fractions. As acknowledged by Post et al. (1986), “The situation appears to be bi-directional: as children learn to order fractions they acquire a quantitative concept of fractions; as they extend their concept of number to include fractions they also learn to order them" (p. 46). So when students understand relative size, they can use important number relationships to solve problems involving rational numbers. To study children's thinking about fractions, it is important to identify the relationships within fractions and connected to physical models that help children develop a quantitative notion of fractions.

CONNECTING PHYSICAL MODELS WITH CHILDREN'S THINKING

The many different approaches to the use of physical models for developing understanding about fractions make characterizing research about them difficult. Researchers have identified and described the strategies that students used to compare and order fractions where some strategies referred to physical models (Behr et al., 1984; Post et al., 1986; Smith, 1995). Some studies examined students' thinking over time as students participated in small group or whole class instruction (Behr et al., 1984; Cramer et al., 2002; Empson, 1999; Streefland, 1991), while other studies have only provided a snapshot of student thinking at one point in time (Clements & Lean, 1994; Smith, 1995; Taube, 1995). There were different methods for introducing and utilizing physical models (Cramer et al., 2002; Peck & Connell, 1991). Some studies have focused on children's thinking while working with only one or two specific physical models (Armstrong & Larson, 1995; Brinker, 1997; Kamii & Clark, 1995) while other studies have included many physical models (Behr et al., 1984; Cramer et al., 2002; Post et al., 1985). Finally, some studies did not provide specific ways to use physical models because their focus was on how children developed their understanding of fraction concepts built on informal knowledge (Empson, 1999; Mack, 1990; Streefland, 1991).

A few studies have examined the impact of specific physical models on children's understanding of comparing and ordering fractions. Brinker (1997) found that many students did not develop an understanding of fraction strips that helped them make a connection with symbolic approaches. Kamii and Clark (1995) and Armstrong and Larson (1995) demonstrated that when students compared fractions represented by a part-whole model, they did not pay attention to part-whole relationships. Instead students used direct comparisons strategies where they attended to features such as just the area shaded or length or number of pieces (Armstrong & Larson, 1995). Using the physical models taught students to identify "equivalent fractions perceptually and figuratively" (Kamii & Clark, 1995, p. 374). Thus, students relied on the visual cues in place of conceptual understanding.

Instead of using physical models, some researchers have suggested that children develop their own models for fractions (Ball, 1993; D'Ambrosio & Mewborn, 1994; Kamii & Clark, 1995). Gravemeijer (1997) indicated that using physical models did not lead to understanding concepts because "these problems stem from passing over the importance of informal knowledge and strategies" (p. 319). Informal knowledge is "applied, real-life circumstantial knowledge constructed by the individual student that may be either correct or incorrect and can be drawn upon by the student in response to problems posed in the context of real-life situations familiar to him or her" (Mack, 1990, p. 16). By providing questions based on real-life situations such as equal-sharing problems, which require dividing items among a certain number of sharers, children build upon their informal knowledge about fractions (Ball, 1993; Empson, 1999; Kamii & Clark, 1995; Streefland, 1991; 1993). As students make their own drawings and decide how to use physical models, they consider which features are relevant in their physical models. Whether children make their own models or use materials provided to them, their understanding of fraction concepts is related to the use of the physical models in both positive and negative ways.

RATIONALE

In 1993, Ball asserted that in the future, “We need more theoretical and empirical research on representations in teaching particular mathematical content… We need to map out conceptually and study empirically what students might learn from their interactions with [representations]” (p. 190). Even though there is an emphasis on using models to teach students about fractions, there is limited research that connects the understanding that students develop about fractions with the use of physical models. For example, the Rational Number Project had a strong emphasis on translations between representations. It focused on children developing conceptual understanding (Behr et al., 1984; Cramer et al., 2002; Post et al., 1986; Post et al., 1985), but did not connect their understanding to specific physical models.

To examine what children learn through the use of physical models, this study focused specifically on comparing and ordering fractions. Equivalency and ordering of fractions are key concepts in the national and state standards, which are repeated and built upon in the elementary mathematics curriculum (National Council of Teachers of Mathematics, 2000; Texas Education Agency, 1997). In addition to equivalency being “one of the most important rational number concepts” (Vance, 1992, p. 263), order and equivalence are also very challenging for students (Kamii & Clark, 1995; Post et al., 1986). Post et al. (1986) proposed that students’ ability to compare fractions and determine equivalence is closely linked with their overall success with fractions. Smith (1995) declared, “If order and equivalence relations are basic to understanding rational numbers and if making correct judgments is more difficult with rational numbers than with natural numbers, then research should emphasize tasks that involve those judgments” (p. 6). Smith (1995) also identified a framework of four perspectives – parts, components, reference point, and transform – that characterize the general approaches that students used for comparing and ordering fractions. Even so, his framework was limited because it focused on students, primarily in middle and high school, who had been learning about fractions in traditional classroom settings and did not examine the role of physical models in these perspectives.

By working with younger students in the classroom setting, this study examined how younger students, beginning to learn about fractions, made judgments about the relative size of fractions. The role of physical models in Smith’s (1995) perspectives was also explored. My study built on the work of other researchers who examined children’s thinking when solving order and equivalency problems by taking advantage of an unusual classroom setting. I conducted the study in a constructivist classroom (Brooks & Brooks, 1999; Carpenter, Fennema, Franke, Levi, & Empson, 1999) where the emphasis was always on understanding mathematics concepts. In this setting, the teacher encouraged students to build on their informal knowledge of fractions (Ball, 1993; Empson, 1999; Streefland, 1991). The teacher introduced some physical models through specific activities; however students made their own models or used any materials available in the classroom when they solved problems and justified their answers. Expanding on previous research, this study 1) connects children's developing understanding of fractions with their use of physical models and 2) identifies relationships that elementary children in a constructivist classroom use to solve order and equivalence problems.

THE STUDY

My study focused on children's understanding of fractions and the role of physical models when third through fifth grade children in a constructivist classroom solved order and equivalency problems. Throughout this study, the term "physical models" is used broadly to include manipulatives, pictures, and drawings. The terms "physical models," "physical materials," "concrete models," "concrete materials," "manipulatives," and "manipulative materials" are used somewhat interchangeably in the research and in articles about teaching mathematics. "Manipulatives," "concrete models" and "concrete materials" refer specifically to objects that students can perform actions on that are purchased or constructed by teachers or students. Manipulatives include base ten blocks, pattern blocks, Unifix cubes and computer generated manipulatives that allow students to manipulate objects on the screen. Physical models also include pictures that are used to represent mathematics concepts and drawings that students make when solving mathematics problems.

To examine how children's understanding of fractions and their use of physical models impacts how they order and compare fractions, I conducted this study in a multi-grade classroom where the teacher encouraged students to think when they solved mathematics problems and to chose when and how to use physical materials. All of the teachers at this school used Cognitively Guided Instruction (CGI) (Carpenter et al., 1999) for teaching mathematics. A CGI class provided a rich learning environment for students because the teacher expected them to solve problems using their own strategies, required to justify their thinking, and encouraged them to construct a relational understanding of mathematics. The teacher facilitated this development by the problems given and questions posed during discussions.

The teacher was the instructional decision maker throughout the study. This contrasts with other studies where a fraction unit was either provided to teachers (Cramer et al., 2002), adapted from a commercially available curriculum unit (Brinker, 1997), or developed with the teacher (Empson, 1999). Using materials, resources and ideas that she had been collecting over several years and by gathering information about children's thinking through observations and discussions, the teacher planned instruction based on the needs of individual students.

A typical lesson in this class included giving the children word problems, and allowing them to solve the problems using one or more strategies by themselves or while working with others. The teacher observed students, asked questions, and brought the class together to share their solutions. Most of the time students used any physical models that they chose, but on some occasions a specific task required using specific physical models, such as when students played games with the fraction strip kits. At the same time individual students developed their own approaches for solving problems with fractions and decided how to use physical models, each student's ideas evolved during class discussions from sharing solution strategies and by working in groups. Kazemi & Stipek (2001,) identified these classroom norms as critical for "promot[ing] conceptual thinking" (p. 64).. Although individual students solved problems using their own strategies, the interactions with their classmates and teacher in the social setting of the classroom likely influenced how they solved problems.

Since fraction concepts are typically introduced and explored in third through fifth grades (National Council of Teachers of Mathematics, 2000; Texas Education Agency, 1997), I chose to work with students in these grades. The 13 participants consisted of four third graders, five fourth graders, and four fifth graders from the same multi-grade class. This was an unusual aspect of my study because many other studies only included one or two of these grade levels (Armstrong & Larson, 1995; Ball, 1993; Behr et al., 1983; Behr et al., 1988; Behr et al., 1984; Cramer et al., 2002). Working with students in three grade levels provided diversity in terms of children's development, partially due to their previous experiences with fractions.

Data collection included clinical interviews and classroom observations. All 13 students participated in video taped clinical interviews before the beginning of the unit and the week after they completed the unit. Eight students also participated in interviews about mid-way through the unit. The interviews included fraction problems with equal sharing, ratios, computation, paper folding, and order and equivalence. By using data from clinical interviews, I examined children's understanding and how they chose to use physical materials from the social arena of the classroom. When students worked on the fraction unit, I videotaped the mathematics class. In addition to capturing how individual students solved problems and explained their solutions to others in mathematics class, this provided information about the structure of the class, social expectations for participation, and the use of physical models.

The primary focus of the analysis was order and equivalence questions presented symbolically and through story problems during the clinical interview. Through this analysis, I identified the relationships to which students attended. These approaches or "perspectives" for comparing and ordering fractions are a major part of the research findings. Through the coding of the interview questions and examination of the patterns within the coding, themes emerged that connect these perspectives to the use of physical models. I analyzed the classroom data for connections between instruction, how children solved problems during the individual interview, and the emerging perspectives and themes.

RESEARCH QUESTIONS

This study examined the relationships that elementary students attended to when comparing and ordering fractions as well as the relationships between students' understanding of fractions and the use of physical models. The guiding questions for this research study included:

> To what relationships do elementary students attend and utilize when comparing and ordering fractions?
>
> How are physical models utilized and extended by elementary students for comparing and ordering fractions in a constructivist mathematics class?
>
> How do children's approaches to solving order and equivalence fraction problems and the use of physical models support the development of number relationships?

SUMMARY

This study examined the relationships on which elementary students focused and the role of physical models when learning how to compare and order fractions. This unusual classroom setting provided an ideal learning environment to examine how students constructed their understanding of fractions by solving problems and using physical models. Due to the teaching philosophy in this class, the teacher allowed and encouraged the children to solve problems using their own strategies and to use the physical models in ways that made sense to them. Between the clinical interviews with individual students and daily classroom observations of their mathematics class, I collected in-depth data about how each child solved problems, justified their answers, and used physical models. Since these students were at different grade levels in the same class, the students provided diversity in age and previous experiences with fractions. Due to the complexity of children's learning about fraction concepts and the unusual setting, these students were not representative of all similarly-aged students. Instead, this exploratory study illustrated how children thought about fractions when they were in a mathematics environment where they developed a quantitative notion of fractions and used physical models to support their understanding.

Chapter 2: Conceptual Framework

The conceptual framework for this study is based on how the use of physical models and children's developing understanding of fraction concepts impact children's approaches to comparing and ordering fractions. Chapter one described how understanding is based on the kind of connections made between different pieces of information (Hiebert & Carpenter, 1992; Skemp, 1978). Since more relationships translate into stronger understanding, identifying and using relationships in mathematics is imperative for developing conceptual understanding of fractions. Throughout this chapter, relationships in the use of physical models and in the strategies for comparing and ordering fractions are explored.

Students need to identify and extend relationships that are represented in the physical models or use relationships to create representations to develop a stronger understanding of fraction concepts. Students need to learn about relationships that are specific to different types of physical models. For example, area, linear, and set models are different representations for fractions and different relationships are important (Behr et al., 1988; D'Ambrosio & Mewborn, 1994; Larson, 1980). When students work with fractions symbolically, they need to examine and build on number relationships. These relationships may be within the numerator and denominator of the same fraction, across numerators or denominators of different fractions, or between unit and composite fractions (Behr et al., 1984; Smith, 1995; Tzur, 1999). To make links between physical models and symbolic notation for fractions, students need to connect number relationships with relationships in physical models (Brinker, 1997; D'Ambrosio & Mewborn, 1994; Streefland, 1991). Students demonstrate this connection when they understand the inverse relationship in unit fractions or use number facts to draw representations. These relationships are important if children are to develop a relational understanding of fraction concepts.

The first portion of this chapter focuses on the research on physical models. I address the different kinds of physical models in the first section and summarize the results of different studies examining the complexities and effectiveness of various physical materials in the following sections. Although researchers tend to agree physical materials are important, I highlighted the differences between their approaches for using the materials. The major heading in this chapter focuses on developing fraction concepts. I describe a “quantitative notion” of fractions since it provides a foundation for comparing and ordering fractions. Then I discuss the different types of order and equivalence problems and how some of the language used in these different problems causes confusion for students. Finally, Smith’s (1995) perspectives describing the general ways children solve order and equivalence problems is used as a structure for examining the strategies found in the literature.

PHYSICAL MODELS

As discussed in the first chapter, I use the term “physical model” to refer to concrete materials that can be acted upon as well as two-dimensional pictures or drawings. Physical models which are commonly utilized for teaching rational number concepts to children are listed in Table 1.

Table 1**:** Physical Models for Fractions

		CLASSES OF PHYSICAL MODELS		
		Area Models	*Linear Models*	*Set Models*
SPECIFIC PHYSICAL MODELS	*Prefabricated*	• circular or rectangular fraction pieces • fraction bars • pattern blocks	• Cuisenaire rods • fraction bars	• counting chips • Unifix cubes • beans
	Constructed	• paper folding • paper cutting • drawing with one item (i.e. cakes, pizzas) to partition	• number line • paper strips • string • drawing lines	• drawing with multiple items that represent a whole
	Dynamic	• Candy Bar microworld	• Sticks microworld	• Toys microworld • Copycat

Rational number models can be classified either as area, linear or set models. Area and linear models are continuous, which means the fractional relationship between part and wholes is dependent on the “size and shape” (Behr et al., 1988, p. 64) of each of the parts. Set models are discrete, and the fractional relationships is dependent upon the “number of items” (Behr et al., 1988, p. 64) out of the whole set. I have also divided the models into three categories: prefabricated, constructed, and dynamic. Prefabricated models are manufactured for classroom instruction and are made of durable materials which can be reused for many years. Constructed models are made by the students or teachers during the course of instruction or through students' explorations. Dynamic models are software programs designed to provide interactive models for rational numbers (D'Ambrosio & Mewborn, 1994; Thompson, 1992; Tzur, 1999).

Table 1 does not indicate whether the teacher or student determines how the materials are used during instruction, it merely provides a description. For example, students may use counting chips to develop their own way to solve a problem or a teacher may direct students to follow specific steps to make a model of fractions using paper folding.

Prefabricated models are designed to demonstrate specific patterns or relationships. For example, fraction bars are designed so the representations for 1/3, 2/6, 3/9 and 4/12 have the same length and area. Fraction bars are placed in the area and linear classes in Table 1 since students may use the fraction bars as a linear model when they compare the lengths or they may consider the area of the shaded sections. Also, the relationships between unit fractions and composite fractions are maintained: 2/6 is the same size as 1/6 twice, and 3/6 is the same size as 1/6 three times. These relationships are part of the construction of the physical models, although students may not be aware of, or focus on, these relationships. A limitation of prefabricated continuous area and linear models is they are divided into pieces that cannot be easily partitioned, so students can manipulate the materials as though they are discrete (P. J. Callahan, personal communication, October 26, 2001). For students to use a prefabricated model to develop their understanding of fractions, they must be able to identify and use the relationships in the model.

As an alternative to prefabricated models, constructed models can be made by students or teachers, but the relationships may be more difficult to create. For example, paper folding can be used as a model to find, make predictions about, and develop a rule for generating a set of equivalent fractions. But this model is limited by the way students make folds, and they can make folds that do not maintain relationships. My conjecture is that the construction of these models must be guided by children's understanding of the underlying relationships in fractions; otherwise the inaccuracy of the model will hinder students' ability to solve problems and limit their understanding of fractions.

Dynamic models offer the advantage of using a computer program developed to "provide constraints on students' concrete actions in places that are likely to draw their attention to relationships among meaning, notation, and expression" (Thompson, 1992, p. 127). For example, when students partitioned a line in the Sticks microworld, the computer program made all of the pieces equal-sized (Tzur, 1999). Although there are benefits to using dynamic models such as Sticks, these programs are not widely available for classroom instruction.

With each type of model there are potential benefits as well as limitations. While prefabricated models offer durability and examples of the most commonly used fractions, they require students to identify and use the relationships embedded in the model. In contrast, constructed models can be rich learning tools if the underlying relationships guide the construction, but they may hinder student's problem solving abilities if the model is constructed inaccurately. Dynamic models can prevent inaccurately constructed models, but the constraints embedded in the program may also limit the relationships students can identify and use. Regardless of which physical models are introduced or utilized in classroom instruction, the relationships in the physical models and how student build on those relationships determine what students understand. Some rational number studies have attempted to assess the complexity of using certain classes of models for developing understanding of concepts.

Comparison of Physical Models

Continuous models are easier for students to use for solving problems than discrete models (Behr et al., 1988; Pitkethly & Hunting, 1996). Behr et al. (1988) explained that students must think differently in solving problems involving these two models. Based on the results presented by Larson (1980) and Behr et al. (1983), students were most successful when the relationships in the physical models were more closely related to the fractions they had to identify.

Norvillis (1976) suggested the most difficult model for students was a number line (cited in Larson, 1980). Of the two models Norvillis studied, students identified proper fractions more successfully with an area model than on a number line. She recognized that children did not pay attention to the scaling on the line and often treated the number line as one unit long regardless of the scale. In a later study, Larson (1980) examined students' ability to identify fractions on a number line and found students were more successful on lines one unit long where the number of pieces was equal to the fraction denominator.

Similar to Larson's (1980) results, Behr et al. (1983) found students were more successful with area models than with linear models. This study examined the perceptual cues that impacted students' ability to identify different fractions in pre-partitioned area, linear, and set models. When the relationships were most similar to the fraction to be identified, students were most successful. For example, students easily identified the pictures partitioned into the same number of equal-sized pieces as in the denominator of the fraction they had to find. Other variations became progressively more difficult as students had to make extra partitions or combine sections because there were fewer or more sections than needed. The most difficult problems did not have a relationship between the physical model and the fraction. For example, when a fraction was divided into thirds, the student had to ignore the original partitions to identify a portion with fourths.

Contrary to the research presented by Larson (1980) and Behr et al. (1983), the Fraction Project used computer microworlds to introduce linear models prior to area models because "linear models help children construct rich definitions of fractions" (D'Ambrosio & Mewborn, 1994, p. 152). These researchers indicated the area model limited children's understanding of fractions by encouraging students to use a counting strategy to determine the number of shade pieces over the number of pieces all together. The inconsistencies between these studies may be attributed to the relationships students observed and used. When researchers presented students with the number line (Behr et al., 1983; Larson, 1980), students focused on the relationship of the part to the whole. By using a dynamic program that allowed students to divide the line and move the sections (D'Ambrosio & Mewborn, 1994), students examined relationships between the parts and later iterated the unit fraction to generate composite fractions. Each of these studies indicated that students identified and used different relationships to solve problems.

Effectiveness of Physical Models

Although physical models are recommended for helping children learn about mathematics, the benefits of using physical models for instruction have not been clearly demonstrated. Reviewing previous research on the effectiveness of using base ten blocks for developing students' understanding or ability with computation, Thompson (1992) found conflicting results. He proposed that these different conclusions may be accounted for by how the students utilized concrete materials, how concrete materials were connected to symbolic representation, and the overall goals of the studies. In addition to learning about relationships between the physical models and symbolic notation, students needed to identify and use relationships included in the construction of the physical models. These relationships impacted what children understood so they could solve novel problems and justify their answers.

The Rational Number Project (RNP) has investigated students' developing understanding of fractions through curriculum which includes a variety of manipulatives (Behr, Herel, Post, & Lesh, 1992; Behr et al., 1988; Behr et al., 1984; Cramer et al., 2002; Cramer et al., 2000; Post et al., 1985). RNP is a major research project that examined the use of physical models in the development of students' conceptual understanding of rational numbers including comparing and ordering fractions. Results from a small-group teaching experiment indicated fourth grade students were developing a quantitative notion of fractions that provided a foundation for order and equivalency (Behr et al., 1984). For students to develop this understanding of fractions, they had to find and build on relationships. For example, students could use manipulatives, but they also recognized important relationships between the numerator and denominator of a fraction and relationships to make comparison to reference points. Students used whole number relationships some of the time when they solved problems; however, it caused some problems when the denominators were not equal. Since these fourth graders still had gaps in their understanding, Behr et al. advocated beginning instruction on fractions in the third grade. Although the researchers identified relationships that students used to solve problems, they did not describe how the use of specific physical models influenced children's understanding.

Another RNP project involved a large-scale implementation study with over 1600 students (Cramer et al., 2002; Cramer et al., 2000). Researchers provided the teachers with the Rational Number Project curriculum and compared the learning of students using this curriculum to the learning of students using commercially-available traditional curriculums. The RNP curriculum emphasized using physical materials such as fraction circles, chips, and pictures in almost all of the lessons. The curriculum also included a comments section for teachers to explain issues related to teaching fraction concepts using these materials and to describe children's thinking when solving similar problems based on previous research. Although the commercial curricula mentioned using physical models for teaching fractions, the textbooks emphasized symbolic proficiency for fractions.

There were statistically significant differences in favor of the RNP students overall and on four subsections of their analysis. One of the subsections that did not have a statistically significant difference was on computation, which surprised the researchers because they anticipated students in the traditional curriculum, which emphasized computation, might outperform the experimental group. When analyzing the qualitative differences from the interviews, the researchers found the RNP students used a conceptual approach to solve many problems while the other students relied on procedures. Not only did the RNP students develop a stronger conceptual understanding of fractions and use mental images more frequently to explain their answers when comparing fractions than students in traditional classrooms, these students were better able to verbalize their solutions strategies. The evidence confirmed the large-scale implementation of the RNP curriculum was successful as measured by student learning on the post-test and interviews.

Peck and Connell (1991) conducted a smaller-scale study comparing two classrooms learning about fractions using very different approaches. In a fifth grade classroom with average students, they used a constructivist approach where students began by using physical models and moved towards more symbolic approaches. In the accelerated fifth grade class, students followed the traditional curriculum. After instruction, the students solved problems and explained their solution strategies. Students in the accelerated class tended to use symbolic reasoning to explain their answers, but they sometimes used inappropriate reasoning such as additive relationships and were unable to answer a non-traditional problem that required a fraction in the numerator. In comparison, the average students who were taught using a constructivist approach, frequently used physical models to justify their answers. Not only were these students better able to explain and justify their answers, they were more successful in answering the non-traditional problem. The success of the average students on the non-traditional problem may be attributed to their ability to build on relationships they understood through their experiences in the constructivist classroom. These observed differences included students' ability to think about and use symbolic procedures, justify answers, actively solve problems, and confidence in approaching new problems.

Other studies have examined how students use concrete materials based on the linear model. For example, Brinker (1997) examined how students used fraction strips to solve problems. She found that the fourth and fifth graders did not see the relationships in the fraction strips, which made it difficult to connect the concrete manipulative to their symbolic strategies. Several studies have found that students using fraction strips or other linear models have a tendency to use an end-to-end approach that results in an estimated answer instead of using equivalent relationships to find exact answers (Brinker, 1997; D'Ambrosio & Mewborn, 1994; Streefland, 1991). Though it is not clear whether the students used the fraction strips as a linear model intended by the instructor or as an area model (D'Ambrosio & Mewborn, 1994), it was clear that students were not using relationships included in the construction of the physical models. D'Ambrosio and Mewborn (1994) endorsed building on the relationship between unit fractions and composite fractions by using a linear model in a computer microworld. The precision of duplicating unit fractions and combining these pieces in the computer microworld led to understanding concepts that were not possible using imprecise paper models.

It is difficult to make generalizations about the effectiveness of physical models in instruction based on these studies. They were conducted in a variety of settings using assorted physical materials. To determine the effect of using physical models, more research needs to examine the specific relationships that children understand based on using physical materials. Some of the variation in results may be accounted for, at least partially, by the underlying philosophy about how to use specific models to help children understand concepts.

Use of Physical Models

Recognizing physical models are, in and of themselves, not enough to teach mathematics concepts, researchers have proposed multiple approaches for how physical models should be used. Many recommended specified steps that can be taken to introduce physical materials (Behr et al., 1988; Bezuk & Bieck, 1993; Connell & Peck, 1993; Peck & Connell, 1991; Post et al., 1985; Thompson & Lambdin, 1994), whereas others provide a framework for considering how the physical materials are utilized (Gravemeijer, 1997; Kent & Gravemeijer, 2001). Gravemeijer (1997; Kent & Gravemeijer, 2001) describes both a “top-down” and a “bottom-up” approach for using concrete models in the classroom. Kent and Gravemeijer (2001) suggest physical models in the classroom can be used “as ‘tools for thought’ or as ‘tools to get an answer’” (p. 25). Gravemeijer and Kent’s distinction provide a useful framework for analyzing the approaches other researchers used with physical models.

Physical Model Framework

The top-down and bottom-up strategies are different ways to connect formal mathematics and physical models (Gravemeijer, 1997). In a top-down approach, the physical models are developed by experts who then introduce the materials to children. Since the experts understand the underlying mathematical concepts, the physical models embody relationships the expert values and wants students to learn. Students are taught how to use the physical models to solve a problem. The intention is by using the physical materials, students will understand the relationships the expert intended, which will aid in connecting the physical models with the symbolic notation. For example, base ten blocks are chosen because they illustrate the relationships between ones, tens, and hundreds in the place value system. Students are shown how to subtract using base ten blocks by starting with the ones and trading as needed and then they learn how to record these steps in the traditional algorithm form (Thompson & Lambdin, 1994). Note that students are using physical materials to solve a problem, but the steps they follow are not connected to their informal knowledge or their previous learning.

To connect students' informal knowledge with formal mathematics, Gravemeijer (1997) advises students need to begin by developing their own models using a bottom-up approach. By presenting problems in real life contexts (Streefland, 1993), this approach builds on intuitive knowledge that children have before instruction (Behr et al., 1992) and requires students to find and build on relationships they understand. For example, Streefland (1991) gave students a problem with children sitting at tables and sharing pancakes, and they had to determine each child's fair share. Then students were given the opportunity to work with physical models, including using drawings, to help figure out the problems. Frequently students used repeated halving to share the pancakes, which was similar to the strategy Brinker (1997) reported. Students were developing their own physical representation, which Gravemeijer called "*models of*" the mathematical phenomena. As students had opportunities to explore similar problems further and understand relationships, they translated their drawings into the ratio tables later in the curriculum unit (Streefland, 1991). At some point, this understanding of the model allows them "to use this model as a *model for* mathematical reasoning" (Gravemeijer, 1997, p. 329).

In using a model *for* mathematical reasoning, students may use physical materials to help them figure out an answer or as a way to think about mathematical relationships. However, in a top-down approach where there is a focus on the procedure, students are likely to use the physical models as "tools to get an answer" (Kent & Gravemeijer, 2001). If students' explanation for how they solve problems is based on the procedures they followed, they are using an analytic argument to support their answer. Since students are primarily concerned about getting an answer, I believe they are not looking for or paying attention to relationships in the physical models. In a bottom-up approach, students may either use the materials to get answers or as "tools for thought." To encourage the use of the materials to further thinking and understanding about concepts, problems must go beyond the available physical materials and require students to provide a substantial argument to justify their answer. Unlike analytical arguments that focus on the procedures, a substantial argument is based on relationships and generalizations. By providing numbers beyond the use of physical materials, students must extend their thinking and use relationships to solve problems.

Recommendations for Instruction

Different researchers have proposed specific ways to introduce and use physical models to develop fraction concepts. Some of the suggestions are generic so they can be connected with either a top-down or bottom-up approach. On the other hand, some of the suggestions tend to favor one approach over the other. Though the authors did not specifically use “tools for thought” or “tools to get an answer,” the recommendations for using physical materials provide enough description to infer a position.

In a summary of the literature, Bezuk and Bieck (1993) emphasized the importance of using a variety of physical models to develop fraction concepts including: fraction kits either prefabricated or made by the teacher, Cuisenaire rods, pattern blocks, colored chips, colored tiles, and number lines of different sizes with different partitions. They posit that by using a variety of models, children develop different understandings of fractions including fractions as part of a whole, part of a group, and a place on a line. Bezuk and Bieck urged only using fractions that *could be represented* using the manipulatives, which was contrary to Kent and Gravemeijer’s (2001) notion to include problems that *could not be represented* using manipulatives. Since students could use the manipulatives to solve problems, it might lead them to use a "tool to get an answer" (Kent & Gravemeijer, 2001). Bezuk and Bieck also indicated students should "use more than one mode of representation in instruction" (p. 124). After students understood a concept using one type of representation, they then represented the concept using a different material which required students to use a "tools for thought" approach to the manipulatives. Through this process, students could compare and contrast the physical materials, which should bring out relationships both within specific models and across different models. In addition, Bezuk and Bieck explained the verbal and written forms had to be connected to the physical materials, always building upon students' understanding developed through the use of the manipulatives. These recommendations could be interpreted in both a bottom-up or top-down style for using physical materials.

Thompson and Lambdin (1994) rejected the top-down approach. Instead of focusing on what children should do with or without physical materials when planning for instruction, Thompson and Lambdin emphasized teachers' first concern must be "What do I want my students to *understand*?" (p. 558). When students were solving mathematics problems, the physical materials provided a hands-on way for students to find answers and a concrete way to describe their solutions to the class. Through working with the materials and talking about both the physical materials and the related mathematics, conversations focused on "constructing strong connections among ways of thinking about concrete situations and conventional mathematical language and notation" (p. 558). Thompson and Lambdin provided some hints for using materials, but always returned to what children should *understand* as a guide for making instructional decisions.

In contrast to the general approaches described previously, Peck and Connell (1993; 1991) provided a specific set of steps for using physical models that followed a bottom-up approach. They proposed posing problems in the beginning for students to engage in while working with the physical materials. The instructor does not tell students specific steps for using the manipulatives, rather students figure out what to do through their active involvement working with the materials to solve problems. The teacher introduces symbols in relationship to students' work with the physical models. Connell and Peck (1993) declared, "As the children used these physical materials to solve problems, they actively constructed operations and principles of arithmetic" (p. 333). As teachers encourage students to move beyond the physical materials, students first draw pictures of the physical materials and then use mental images of the physical materials. This encourages the "tools for thought" approach as students solved problems not directly tied to the physical materials. Through these steps, Peck and Connell believed students could reach the fifth level where "students construct strong arithmetic generalizations and problem solving skills" (Peck & Connell, 1991, p. 4).

The Rational Number Project encouraged a top-down approach. In curriculum materials developed for research purposes and for classroom use, the instructor introduced students to a variety of concrete manipulatives such as fraction circles, rectangles, Cuisenaire rods, paper folding, number lines, pictures, and counting chips during instruction (Behr et al., 1988; Behr et al., 1984; Cramer et al., 2000; Post et al., 1985). Most of the lessons required students to use the manipulatives. Since the curriculum guide for teachers included directions explaining "how to use concrete materials to model fractions" (Cramer et al., 2000, p. 8), this program encouraged a top-down approach for teaching fraction concepts.

Similar to the proposal by Bezuk and Bieck (1993), RNP emphasized making translations with concrete manipulatives (Behr et al., 1992; Behr et al., 1983; Behr et al., 1988; Behr et al., 1984; Cramer et al., 2002; Cramer et al., 2000; Post et al., 1985). There were two types of translations students were expected to learn. One kind of translation was between different manipulatives. This type of activity could lead to a bottom-up approach for using physical models, yet descriptions indicate the translations were taught in a top-down manner. Behr et al. (1988) suggested that after students became skilled with one particular manipulative, the teacher should introduce a second manipulative and guide the class in a discussion about how the teacher used the manipulative, how they could use it themselves, and how it was similar to and different from the previously used manipulative. When students are using a procedure modeled by the teacher, they may be inclined to use the manipulatives as "tools to get an answer" and not identify or use relationships. At the same time, answering questions about the similarities and differences may encourage students to use the manipulatives as "tools for thought" as they look for relationship between and within the materials.

The other type of translation required moving between the five forms of representations: manipulatives, pictures, real-life examples, written symbols, and spoken language (Behr et al., 1983; Behr et al., 1988; Behr et al., 1984; Cramer et al., 2002; Cramer et al., 2000; Post et al., 1985). This emphasis on translations encouraged students to use the manipulatives as "tools for thought" as they made identified relationships or connections across a variety of representations and moved fluently between the different representations. When students depended on the physical materials, Behr et al. (1984) declared students needed more time working with the manipulatives. When students solved problems without the manipulatives, they recommended that students "should not be required to use them" (p.338).

There are a variety of ways for students to use manipulatives to develop their understanding of fraction concepts. Making connections between physical materials and symbolic notation is a part of developing fraction concepts. The Rational Number Project emphasized translating between real-life examples, physical representations including manipulatives and pictures to the oral words and written symbols (Behr et al., 1983; Behr et al., 1988; Behr et al., 1984; Cramer et al., 2002; Cramer et al., 2000; Post et al., 1985). This is similar to the approach described by Bezuk and Bieck (1993). In a more limited manner, Peck and Connell (1993; 1991) stressed symbolic notation must always be connected to representations. Though there are many opportunities for students to seek out and extend relationships in the physical models, not all students make these connections.

Concerns about Connecting Physical Models to Symbolic Notations

Even though the intention in many of the previously mentioned studies was to make connections between physical models and symbolic notation, researchers have documented the disconnect between materials and symbols in students' learning (Brinker, 1997; D'Ambrosio & Mewborn, 1994; Hope & Owens, 1987). Students were sometimes satisfied with two different answers that they found using manipulatives and symbols (Brinker, 1997; Hope & Owens, 1987) or students switched to using symbols to solve problems when they recognized the manipulatives only gave an approximate answer (Brinker, 1997). Furthermore, students did not try to connect their symbolic approach to the fraction strips, and a student who relied on the fraction strips was unable to use symbolic strategies to find exact answers (Brinker, 1997). As described by Hope and Owens (1987, p. 36), "these children see little connection between the worlds of the physical and symbolic. Each is viewed as a different system operating independently" (p.36).

Some physical models allow students to work without making connections to the representations. In one study where students were using pattern blocks, they did not refer to the fractional relationships in their descriptions of the various ways to make hexagons (D'Ambrosio & Mewborn, 1994). Instead students talked about the different pieces by color, such as "I used two reds" or "One red, one blue, and one green" (p. 156). D'Ambrosio and Mewborn hypothesized that the pattern blocks did not fit with the teacher's way of discussing fractions using a part-whole approach where the denominator referred to the number of pieces and the numerator referred to how many pieces were shaded. So although the researchers intended for certain fractional relationships to be explored through these materials, the teacher missed the opportunity to make connections between the representations with the pattern blocks and fraction terminology.

Summary of the Use of Physical Models

Although the top-down and bottom-up strategies are very different, these approaches provide opportunities for students to connect physical materials with formal mathematics (Gravemeijer, 1997). Examining whether students use the physical materials as “tools to get an answer” or as “tools for thought” provides another framework to view approaches for using manipulatives (Kent & Gravemeijer, 2001). By using a bottom-up approach to introduce physical models and expecting students to use them as “tools for thought,” students are required to focus on relationships. These relationships are extremely important when students are constructing their own models.

In my study, the teacher introduced physical models such as two-colored counters, pattern blocks, and fraction strip kits through specific activities. Coming from an underlying CGI philosophy for teaching mathematics, she used a bottom-up approach where she expected students to solve problems in ways that made sense to them. Therefore, students chose how and when they wanted to use physical materials to solve problems or justify their answers based on the relationships in the physical models and in the symbols that they understood and were most relevant to them.

Student Constructed Models

Although many studies have examined the use of models introduced by the teacher for helping children develop understanding of fraction concepts, other researchers have questioned the benefit of this approach (Ball, 1993; D'Ambrosio & Mewborn, 1994; Kamii & Clark, 1995). Should physical models be introduced? If so, how? If not, how do children develop their own models of rational numbers? Several researchers have put forth alternatives or have developed instructional units that are not dependent on concrete materials (Empson, 1999; Kamii & Clark, 1995; Streefland, 1993).

Ball (1993) asserted that the teacher must decide whether to introduce manipulatives or have students develop their own pictorial representations. One advantage of allowing students to develop their own drawings is students must address issues related to rational number concepts. For example, do rectangles have to be the same size to compare fractional amounts? Children must also consider whether pieces must be the same shape to be equivalent (D'Ambrosio & Mewborn, 1994; Kamii & Clark, 1995). These issues require students to develop their conceptual models of rational numbers. Students often believe equivalent fractions require finding shaded sections that "look alike" (D'Ambrosio & Mewborn, 1994, p. 154). In a study comparing two identical rectangles cut in half two different ways (across the middle and diagonally), students used perception about the relative sizes of the pieces to determine they were not equal to other (Kamii & Clark, 1995). Ball (1993) observed a limiting definition of fractions by students who started to associate a visual representation of a fraction with a certain shape. For example, students started to draw a quarter of a circle to always represent one-fourth. Instead of giving students manipulatives, Kamii and Clark (1995) advised that students should develop their own models based on their informal knowledge and advocated approaches such as fair sharing demonstrated by Streefland (1991; 1993) and partitioning demonstrated by Mack (1990).

Partitioning wholes into equal-sized pieces builds on children's informal knowledge (Behr et al., 1992; Charles & Nason, 2000; Lamon, 1996; Mack, 1990; Y. Pothier & Sawada, 1983; Saenz-Ludlow, 1994; Smith, 1995; Steffe & Olive, 1991). When students make their own drawings and use partitioning, the relationships they use are important. Pothier and Sawada (1983) described the development of partitioning strategies in young children. At the first level, children focused on creating a given number of pieces. Although they began to use halving, they were not concerned about equal-sized pieces. A second level was when students used repeated halving to make halves, fourths, eighths and sixteenths, but equal-sized pieces were still not important. They also were aware of the relationships occurring through repeated halving. The third level was more advanced because students were concerned about making equal-sized parts, but they could only do so for the limited fractions that could be obtained by repeated halving. At the fourth level, students recognized using repeated halving only allowed them to create certain fractions, so they began by making partitions that did not create two equal-sized pieces. For example, they began with an off-centered partition for rectangles and the radius of a circle to make thirds or fifths. Students started to look for other relationships to help with partitioning, but the relationships are most apparent in the hypothetical fifth level called "composition." Pothier and Sawada (1983) conjectured that students would use multiplicative relationships to make partitions for composite numbers. For example, students could make sixths either dividing halves into three pieces or by dividing thirds in half. Being able to think about these multiplicative relationships allows children to find equivalent fractions by adding or removing partitions (Steffe & Olive, 1991).

Streefland (1991; 1993) described upper elementary children's learning and understanding of fractions and ratios during participation in a Realistic Mathematics Education (RME) curriculum unit where, "Attention is especially paid to the production of *symbols, diagrams and (visual) models*" (1991, p. 21). Problems were based on real-life examples such as people sharing a limited number of items where the researcher expected students to construct their understanding. Streefland posited that students' ability to progress in their work with fractions and ratios was directly based on their experiences of developing a conceptual understanding using concrete models. He advised that fractions should be initially introduced by fair sharing problems which require children to evenly distribute items (i.e. candy bars, pizza) to a certain number of students.

Based on a similar philosophy for teaching mathematics, Empson (1999) examined how first grade students developed their understanding of fractions over the course of a five week unit that utilized equal sharing problems. Students were not introduced to specific manipulatives, but could use interlocking cubes, paper cutouts, or writing on paper. One of the primary focuses was the models or notations students used and adapted over the course of the unit, which was captured partially by students' oral explanations. By combining tasks built on children's informal knowledge and a classroom environment where the teacher encouraged children to share their ideas, students learned fraction concepts as indicated by the individual interviews. These first graders understood basic fraction concepts beyond repeated halving and had a limited understanding of equivalence. Beginning with students' informal knowledge and allowing students to build their own physical models for solving problems provides an alternative approach for using models to develop fraction concepts.

Summary of Physical Models Research

Researchers have examined and evaluated different physical models. Although there is some agreement on what kinds of models are more accessible and which ones are more challenging for students (Behr et al., 1983; D'Ambrosio & Mewborn, 1994; Larson, 1980), there are multiple approaches for how to actually use the physical models with students (Behr et al., 1984; Bezuk & Bieck, 1993; Cramer et al., 2002; Peck & Connell, 1991; Thompson & Lambdin, 1994). How physical models are used can range from teachers telling students what to do with the materials (Cramer et al., 2002) to letting students develop their own models through exploration and real-life problem situations (Empson, 1999). For any of these approaches to be effective, students must build relationships by using the physical models that further develops their "quantitative notion" of fractions (Post et al., 1986).

DEVELOPING FRACTION CONCEPTS

We want children to understand various characteristics of rational numbers so that they can develop number sense with fractions. Post et al. (1986) term this understanding as "a quantitative notion of rational number" (p. 40). Instead of viewing fractions as two whole numbers with a line in-between, students have to learn "rational numbers are numbers" (Post et al., 1986, p. 40). Children must understand there are multiple ways to represent the same rational number such as by a fraction, decimal, number line, percent, or ratio (Markovits & Sowder, 1991; Post, Behr, & Lesh, 1982; Post et al., 1986; Stone, 1996; Vance, 1992). They must also learn changing the name of the rational number does not change its size (Vance, 1992), and the best representation depends on the specific problem and situation (Markovits & Sowder, 1991; Vance, 1992). To compare and order fractions, children have to learn strategies that are different from the ones they use with whole numbers (Post et al., 1986). As children learn about fractions, they must learn there are an infinite number of rational numbers between any two natural numbers, which does not allow counting or determining the next rational number as with natural numbers (Post et al., 1986; Post et al., 1985; Smith, 1995). Children must consider what the whole is in a particular situation (D'Ambrosio & Mewborn, 1994) and recognize the relationship between absolute and relative size is important when comparing and ordering rational numbers (Post et al., 1986). Children must also develop an understanding that the relationship between the numerator and denominator is significant in comparing and ordering fractions (Post et al., 1986). All of these characteristics form a complex web of ideas that are important for understanding fractions as well as comparing and ordering fractions (Behr et al., 1992; Post et al., 1986).

The remainder of this chapter examines how children develop their understanding of fraction concepts specifically related to order and equivalence. The types of questions that have been used to study students' strategies for, and understanding of, comparing and ordering fractions will be described first. The following section focuses on the strategies students use for comparing and ordering fractions.

Order and Equivalence Problems

Order and equivalency problems require students to consider the relative size of two or more fractions. Even though the terms order and equivalency are often used together and some problems focus on both aspects of comparing fractions, some problems focus only on one or the other. When students are given two non-equivalent fractions and asked which one is larger or smaller, the task only focuses on ordering (Behr et al., 1984; Cramer et al., 2002; Peck & Connell, 1991). This can be extended to include three fractions where the smallest and largest fractions can be identified from the set (Vance, 1986). By using equivalent and non-equivalent fractions within a set of problems and asking students to determine if the fractions are equal or which one is greater, both order and equivalency are addressed (Behr et al., 1984; Smith, 1995). Order and equivalency can also be examined by using real-life problems that ask students to compare two groups with different numbers of children sharing different amounts of the same item and decide where a child receives the larger share (Baker, 1994;

Empson, 1999; Mack, 1990). As long as the ratio relationship between the number of children and items is the same for both groups, a child at both tables gets the same amount. Otherwise, a child in one group will get more than a child in the other group. Other problems focus only on equivalency. Cramer et al. (2002) asked students to name two fractions for each pictorial representation; however, the more common method for studying equivalence is the missing value problem (Baker, 1994; Behr et al., 1984; Empson, 1999; Peck & Connell, 1991). Given one fraction and a second fraction with either the numerator or denominator missing, the student has to determine the missing value that will make the two fractions equivalent.

To solve these problems successfully, students need to understand the language used to discuss comparing and ordering fractions. Confusion arises with terms such as greater, more, fewer, less, amount, and size (Post et al., 1986; Post et al., 1985). Students may question whether to examine the size of each piece or the amount (number) of pieces. When asked which fraction is greater, they need to know if teachers and researchers are asking which size piece is greater or which set of pieces cover more area. Students may not know if the question is asking if there is more area or more pieces when they are comparing fractional amounts. Another confusing term is what is meant by “fair.” Children may conclude fair means "Everybody gets a piece of approximately the same size" (D'Ambrosio & Mewborn, 1994, p. 154). This definition does not indicate that each piece must be exactly equal. Although adults may have a clear understanding of what these questions ask, it may not be clear for students who hear two possible questions with two different answers. Despite these concerns about how children may understand the questions regarding comparing and ordering fractions, researchers have identified common strategies students use to solve these different types of problems.

Strategies for Comparing and Ordering Fractions

The framework for this study to examine children's thinking about fractions when they made comparisons is based on the work of Smith (1990; 1995). Smith identified four perspectives students used for solving comparing and ordering fractions: Parts, Components, Reference Point, and Transform. He explained each perspective was "a distinct and relatively general way to conceptualize and reason with fractions and rational numbers" (Smith, 1995, p. 15). These four perspectives provide an overarching framework for describing how students solve order and equivalence problems and are based both on how students described their solution strategies and their actions in solving problems.

Underlying the perspectives framework are the issues related to the "quantitative notion of rational numbers" proposed by Post et al. (1986, p. 40) and the strategies students use to solve order and equivalence problems. While students are learning about fractions, they must also learn about features of rational numbers crucial for understanding the relative size of different fractions. Researchers have identified a variety of strategies students use for solving order and equivalence problems (Armstrong & Larson, 1995; Behr et al., 1984; Cramer et al., 2002; D'Ambrosio & Mewborn, 1994; Moss & Case, 1999; Post et al., 1986; Post et al., 1985), Smith's (1990; 1995) list of strategies are the most detailed. In addition to identifying the strategies, Smith (1990) compared his list of strategies to widely available textbooks to differentiate between strategies probably resulting from instruction versus ones constructed by students. After identifying these strategies, he grouped the strategies into the four perspectives. Each of these perspectives with related strategies are described in detail in the following sections.

Parts Perspective

The parts perspective refers to using relationships between the whole and the parts to make comparisons. Smith included "the dual constraints of equal size and exhaustion of the whole" in his description of the parts perspective (p. 16). When children begin to learn about fractions, they are often taught fractions using a part-whole model where they count the total number of sections, the number of shaded sections and record the result as two whole numbers with a line in-between (Ball, 1993; Behr et al., 1992; D'Ambrosio & Mewborn, 1994; Mack, 1990; Saxe, Gearhart, & Seltzer, 1999). This approach for teaching fractions may have impacted Smith's finding that elementary students preferred this perspective for solving problems, especially in comparison to the middle and high school students who were very successful at comparing fractions. Although this perspective is easy to observe with physical representations including manipulatives and drawings, Smith included mental models where students did not use physical materials but referred to the parts and the whole in solving problems.

Comparing the amount of area covered by each fraction is a common strategy for comparing and ordering fractions (Armstrong & Larson, 1995; Behr et al., 1984; Smith, 1995; Steffe & Olive, 1991). This was typical of students who had pictures presented to them (Armstrong & Larson, 1995), physical manipulatives to represent the fractions (Behr et al., 1984), their own drawings of the fractions (Behr et al., 1984), or mental images of the fractions (Smith, 1995). Researchers have labeled these strategies for comparing fractions using different terms such as “manipulative” (Behr et al., 1984), “draw diagrams” (Smith, 1995), and “compare models” (Smith, 1995). In their study, Armstrong and Larson (1995) identified two strategies that students used representations to make comparisons: direct comparison and part-whole.

Armstrong and Larson (1995) examined the strategies students used when presented with pictures of rectangles with different portions shaded and asked to decide which one had more shaded. When students used the direct comparison strategy to comparing fractions, they sometimes focused on the shaded area of the rectangle while ignoring the relationship of the part to the whole. This research is supported by other findings that students made judgments about relative size based on perceptual cues (Kamii & Clark, 1995) and students believed the parts had to be the same size and the same shape to be equivalent (D'Ambrosio & Mewborn, 1994; Kamii & Clark, 1995). Instead of paying attention to the area for a direct comparison, some students were concerned about “the length in one dimension, the number of shaded or unshaded parts, [or] a combination of physical features” (Armstrong & Larson, 1995, p. 11). Only considering the number of parts was a strategy observed in other studies as well (Ball, 1993; Mack, 1990). This strategy was more common with the younger students (Armstrong & Larson, 1995).

Students using a direct comparison strategy primarily used the pictorial representations to help solve problems; whereas students using a part-whole strategy verbalized the relationships they observed (Armstrong & Larson, 1995). The "part-whole" strategy was used by older students who tended to make judgments based on the relationship between the parts and wholes. When students referred to the amount shaded as a fraction and compare the wholes, they used the most complex part-whole strategy. For example, students decided two pictures were equivalent, "Because they both take up three-fifths of a cake and the cakes are the same size" (p. 9). Another approach was to still use fraction terminology but not address some of the relevant information such as the size of the wholes. Students decided two pictures were equal because both had 1/2 shaded even though the wholes were different sizes. These students did not understand the importance of absolute versus relative size of the whole (Post et al., 1986). Instead of using fraction terminology to describe the shaded portion, some students compared fractions by focusing on "how the whole was divided and how many parts were shaded" (Armstrong & Larson, 1995, p. 9). A student determined two representations were equivalent because both cakes had five pieces and three were shaded. Again, some students considered the size of both the parts and the whole whereas other students were not concerned about the size of the whole.

Students also compared fractions using a residual approach – examining the complement or missing part of the fraction (Behr et al., 1984; Cramer et al., 2002; Post et al., 1986; Smith, 1995; Zeman, 1991). Although students used a direct comparison strategy of what was missing (Armstrong & Larson, 1995), students also used this strategy to mentally compare fractions (Cramer et al., 2002; Post et al., 1986; Smith, 1995; Zeman, 1991). For example, students compared 2/3 and 3/4 by focusing on the complements 1/3 and 1/4 respectively (Smith, 1995). Since 3/4 has the smaller piece missing, it must be larger.

Paying attention to the relationships in the drawings and manipulatives helped students move towards more advanced strategies that were less dependent upon the physical representation. Bezuk and Bieck (1993) recommended that students use manipulatives to create lists of equivalent fractions and then examine the patterns to help develop procedures for transformations. Post et al. (1985) described how problems requiring students to use manipulatives assisted a child's development from material dependent to material independent. Initially the student used a direct comparison strategy with manipulatives to compare 3/5 and 6/10. Later in the unit, the child answered a question without touching the manipulatives by listing several equivalent fractions to 1/2 for a specific problem, but he only included fractions that could have been made with the materials. He demonstrated that he could plan ahead and purposefully chose specific manipulatives to solve problems without guessing and checking. By the end of instruction, the child solved the missing addend problem 3/4 = 9/□ by building on the multiplicative relationship between 3 and 9: "Three goes into nine three times, and four goes into twelve three times" (Post et al., 1985, p. 31). Through activities using manipulatives and drawing pictures, students build their understanding of fractions.

In addition to using either physical or mental models to make part-whole comparisons, Smith (1990; 1995) identified several other strategies students used based on the parts perspective. Focusing on the number of pieces in the whole or the number of pieces selected allowed students to compare fractions based on the numerator, denominator, or both. Though Smith included these other strategies in both the parts and components perspective, I will describe these other strategies in the components perspective where I believe these strategies fit better with the other literature on children's thinking and approaches for comparing and ordering fractions.

Components Perspective

When students had a parts perspective, they focused on the relationship between the whole and the parts (Smith, 1995). In a components perspective, students focused on the natural number relationships either within the same fraction or between numerators (or denominators) of different fractions. The strategies Smith identified under the components perspective all required the students to make comparisons based on numerators only, denominators only, or a combination of numerators and denominators.

One strategy was for students to consider only the numerator or denominator and apply whole number relationships, attending to the cardinality (size) of the numbers or the ordinal (counting sequence) of the numbers, to determine which fraction was larger (Post et al., 1986; Post et al., 1985; Smith, 1995). For example, a student using this strategy may explain 1/5 is greater than 1/4 because 5 is greater than 4. Students who compared fractions based on the cardinal or ordinal aspects of whole numbers used an invalid strategy that occasionally resulted in a correct answer, but more often caused difficulties for the children (Behr et al., 1984; Smith, 1995).

The only time when whole number relationships are efficient and accurate for comparing the size of the fraction is when the denominators are the same (Behr et al., 1984; Cramer et al., 2002; Post et al., 1985; Smith, 1995). Students can easily determine the fraction with the larger numerator is the larger fraction. For example when comparing 3/8 and 5/8, it is easy to examine the relationship between the numerators to identify the larger fraction. When the numerators are the same, but the denominators are different, students need to consider the inverse relationship between the number of pieces in a whole and the size of the piece (Post et al., 1985). With the fractions 3/4 and 3/5, students can reason that the fraction with the smaller denominator (3/4) has larger sized pieces and is larger overall since there are the same number of pieces (Behr et al., 1984; Cramer et al., 2002; Smith, 1995). Using a combination of the numerator only and denominator only strategies, students reasoned a fraction with a smaller denominator and larger numerator (e.g. 4/7) was larger than the comparison fraction (e.g. 3/8) because each piece was larger and there were more pieces (Behr et al., 1992; Smith, 1995). When the relationship between the numerator and denominator was not as obvious, students considered the relative effects of the numerator and denominator (Smith, 1995). This strategy required students to use qualitative reasoning to weigh the effects of the size of each piece versus the number of pieces (Behr et al., 1992). Using this strategy, a student reasoned 6/10 was larger than 7/15 because one more fifteenth did not negate the effects of the tenths being larger than fifteenths. Even though this strategy may work, the ability to use qualitative reasoning is limited to specific numbers and may lead to incorrect solutions.

Another common strategy students frequently tried was based on additive instead of multiplicative relationships (Behr et al., 1984; D'Ambrosio & Mewborn, 1994; Kaput, 1994; Post et al., 1986; Smith, 1995; Wearne-Hiebert & Hiebert, 1983). Students decided two fractions were equivalent if there was a constant difference within the numerator and denominator or across numerators and across denominators (Smith, 1995). By focusing on additive relationships within the same fraction, students could decide 2/3 and 3/4 were equal because the denominator was one more than the numerator for both fractions. If they focused on additive relationships between fractions, students might notice they could get 3/4 by adding one to both the numerator and denominator of 2/3. Students could use this reasoning to decide either 3/4 was larger because they had to add one more or decide the fractions were equal because they added the same amount. When students used additive relationships, they could either get the wrong answer or get the right answer with an inappropriate justification.

When students focused on multiplicative relationships either within each fraction or between fractions, they were more likely to find a correct answer

(Smith, 1995). Using multiplicative relationships was facilitated by certain number combinations, especially when the relationship could be determined by multiplying or dividing by a whole number. For example, students examined the relationship between the numerator and denominator within a fraction, such as 3/12 and 4/16, and determined they were equivalent because the denominators were both four times larger than the numerators (Smith, 1995). Other times students examined the relationship across the numerators and denominators to find equivalent relationships. They recognized the equivalent relationship between 2/3 and 4/6 because both the numerator and denominator were doubled. In addition, students looked for ratio relationships across numerators and denominators such as 9 is three times more than 3 and 12 is three times more than 4 so 3/4 and 9/12 must be equal (Behr et al., 1984). When students were unable to find a whole number multiplicative relationship, they sometimes digressed and used additive strategies (Smith, 1995).

Reference Point Perspective

When students compare fractions to a third number such as 0, 1/2, or 1, they have a reference point perspective (Smith, 1995). This approach for comparing and ordering numbers is referred to as a “reference point,” “benchmark” or “transitive” strategy (Behr et al., 1984; Cramer et al., 2002; Cramer et al., 2000; Post et al., 1986; Reys, Kim, & Bay, 1999; Smith, 1995). Students may be able to order fractions because one is smaller than one-half and another is larger than one-half (Smith, 1995). Since 7/15 is less than 1/2 and 6/10 is greater than 1/2, 6/10 must be greater than 7/15. Students can decide which fraction is further away from or closer to 1/2. For example, 3/4 is larger than 2/3 because 3/4 is 1/4 more than 1/2 and 2/3 is only half of a third (or a sixth) away from 1/2. These strategies could also be used to make judgments based on how close or far away numbers are from zero or one (Smith, 1995). In contrast to the components perspective where students viewed the components as natural numbers, Smith (1995) claimed students thought of fractions as a rational number when they used the reference point perspective.

Although using reference point strategies is well documented in the literature, it is not a strategy commonly included in mathematics textbooks (Smith, 1990). In one study, elementary students did not use this perspective even though middle and high school students did (Smith, 1995). Some mathematics educators have recommended that this strategy should be included in mathematics instruction and have provided some specific methods for implementation (Reys et al., 1999; Van de Walle, 1998). Since this strategy is only effective under specific circumstances, students must be able to distinguish problems that can be solved using benchmarks. When students identify the situations when this is a useful strategy, they are developing and using number sense to determine relative size.

Transform Perspective

The transform perspective is largely based on the strategies emphasized in instruction in traditional classes, transforming the fractions into equivalent fractions or to decimals (Smith, 1995). Students multiply or divide by fractions in the form of n/n to generate equivalent fractions with a common numerator or denominator, and then they use specific strategies that have been described previously. Although converting fractions to common numerators is not taught in schools, some students used this approach (Smith, 1995). Cross multiplication is another commonly taught strategy for comparing two fractions (Peck & Connell, 1991; Smith, 1995). Although solving problems symbolically can allow comparison of fractions for a variety of problems, Smith (1995) found competent students tended to use transformation strategies when the fractions could not be easily compared using strategies from the other perspectives.

Performing transformations does not mean students understood why the procedures resulted in equivalent relationships (Vance, 1992; Wearne-Hiebert & Hiebert, 1983). Though Smith (1995) identified students as competent in reasoning about fractions, many of the middle and high school students were not able to justify the transformations they used to compare fractions. Students who justified the transformation often relied on their previous experience with partitioning concrete and pictorial representations. Sixth graders tended to believe their answers obtained by manipulating symbols rather than their informal knowledge when they arrived at two different answers (Mack, 1990). At other times, students thought it was acceptable to have one answer from the symbolic transformation and a different answer from using manipulatives (Brinker, 1997; Hope & Owens, 1987).

Although transforming fractions is a viable approach to solving many problems, students who do not understand the transformations make mistakes and disregard answers they know are reasonable. Post et al. (1982) expressed their concern that, "Children are expected to operate at the abstract/symbolic level too often and too soon" (Post et al., 1982, p. 60). In addition, Mack (1990,) claimed that, "knowledge of rote procedures frequently interfered with students' attempts to build on their informal knowledge" (p. 16).

Informal Knowledge

Although Smith (1990; 1995) did not address informal knowledge in his perspectives, children’s informal or intuitive knowledge can be a powerful basis for learning about fractions (Baker, 1994; Behr et al., 1992; Empson, 1999; Gravemeijer, 1997; Mack, 1990; 1991; Streefland, 1991, 1993). Informal knowledge is "applied, real-life circumstantial knowledge constructed by the individual student that may be either correct or incorrect and can be drawn upon by the student in response to problems posed in the context of real-life situations familiar to him or her" (Mack, 1990, p. 16). As pointed out by Mack's definition, children may have incorrect preconceptions about fractions. For example, children used the term "one-half" to refer to a piece of any size (Ball, 1993; Wearne-Hiebert & Hiebert, 1983) or thought the pieces do not always have to be equal (Clements & Lean, 1994). Even though students may have some misunderstandings, their informal knowledge can be an important foundation for formal instruction about fractions (Empson, 1999). Informal knowledge primarily develops a student’s understanding of what is a fraction – a necessary foundation to be able to compare and order fractions.

Instruction providing opportunities for students to use a bottom-up approach for working with physical models allows students to build relationships based on their informal knowledge (Gravemeijer, 1997; Kent & Gravemeijer, 2001). One of the challenges of instruction has been connecting children's informal knowledge and strategies with the symbolic, written form (Behr et al., 1992; Clements & Lean, 1994; Mack, 1990; Steffe & Olive, 1991). Students solve problems using partitioning strategies but then describe the solutions in terms of whole numbers (Ball, 1993; Mack, 1990). Though children use informal knowledge to solve real-life fraction problems such as comparing different numbers of people sharing pizza and determining who gets more, they are unable to do similar problems presented in symbolic form (Clements & Lean, 1994; Mack, 1990). These challenges begin to highlight the importance of students making connections between all of the perspectives and their informal knowledge when they are developing a quantitative notion of fractions.

Competent Students

Smith (1990; 1995) identified competent students based on their ability to solve order and equivalency problems correctly and use conceptual reasoning to explain their answers. Even though he interviewed students in elementary, middle, and high school during this study, all of the competent students were in eighth grade or above. Students who were competent moved between different perspectives in a reflective way to solve problems efficiently. They identified the appropriate situational strategy, such as using benchmarks or comparing residuals, which was only useful for comparing certain fractions. Since each strategy was specific for different kinds of numbers, the types of relationships students needed to be aware of and use varied with the different numbers. They selected general strategies, especially transformations, when the specific strategies were not applicable. Competent students had developed a quantitative notion of fractions that allowed them to solve problems by moving fluently between different perspectives to choose appropriate strategies.

Comparing and Ordering Fractions Summary

As students develop their understanding of rational numbers, there are multiple ways students compare and order fractions. Smith's (1995) four perspectives provide a framework for examining students' thinking and strategies. Within each of these perspectives, students approached problems using different relationships. Using a parts perspective, students focused on the relationship between the parts and the whole. The components perspective required students to examine relationships linking numerators and denominators in the same fraction or linking numerators and denominators in different fractions. When students used relationships involving another fraction such as 1/2, they had a reference point perspective. Students also had to use relationships when they had a transform perspective, but there was more emphasis on the procedures than the relationships. Where Smith's perspectives fall short is in not addressing students' informal knowledge that others have shown to be a viable approach for helping children make sense of fractions (Ball, 1993; Behr et al., 1992; Empson, 1999; Mack, 1990; Streefland, 1993). Also he did not connect the perspectives to the use of physical models which are recommended for teaching fractions (National Council of Teachers of Mathematics, 2000). All of these perspectives and students' informal knowledge are important if we want students to be able to build number sense with fractions to solve problems.

CONCLUSION

Smith's (1990; 1995) identification of the four perspectives used by primarily older students provides a useful framework for analyzing how children solve order and equivalence problems. As students learn about fractions and begin to form a quantitative notion of fractions, they start thinking about the relative size of different fractions. Research has documented numerous strategies students use for comparing and ordering strategies. Although some of the strategies are based on whole number reasoning and are not mathematically correct, many other strategies are mathematically valid and demonstrate students' emerging understanding of fraction concepts. Some of the strategies depend upon, or are directly related to, physical models introduced to students in instruction. With so much emphasis on using physical models, researchers have examined the use of various types of models and different ways of integrating the physical models into mathematics instruction. By focusing on individual students, this study investigated the relationships children developed for comparing and ordering fractions and the role of the physical models in their solutions.

Chapter 3: Methods

I divided the reporting of the methodology into five primary sections: quality criteria, sample, setting, data collection, and data analysis. The first section provides an overview of the criteria for ensuring and evaluating the quality of qualitative research. I addressed general techniques used throughout the study and additional techniques within the other sections as appropriate. The second section describes the sample. In the third section, I described the school and classroom environment, mathematics instruction, and the use of physical models. The last two sections focus on data collection and data analysis. Data collection consisted of a combination of classroom observations and individual clinical interviews with the students. By working with and listening to the students in a problem-solving mathematics class and conducting individual clinical interviews, I examined children's thinking about fractions and their use of physical models when comparing and ordering fractions.

Quality Criteria

I conducted this qualitative research study from an interpretivist paradigm, using constructivist learning theory from psychology. Fosnot (1996) explains that constructivism "construes learning as an interpretive, recursive, building process by active learners interacting with the physical and social world" (p. 30). Throughout the course of this study, I took many steps to ensure the quality of my findings.

Trustworthiness is the primary aspect of quality frequently mentioned in the qualitative methodology literature (see Creswell, 1998; Ely, Anzul, Friedman, Garner, & Steinmetz, 1991; Erlandson, Harris, Skipper, & Allen, 1993; Glesne, 1999; Lincoln & Guba, 1985; Miles & Huberman, 1994). Using qualitative research terminology, Schwandt (1997, p. 164) defines trustworthiness "as that quality of an investigation (and its findings) that make it noteworthy to audiences." After describing the four features of trustworthiness, I address some of the specific methods used throughout this study. In the remainder of the chapter, I describe specific techniques used to ensure the trustworthiness of this study's results.

Trustworthiness

Lincoln and Guba (1985) proposed credibility, transferability, dependability and confirmability as four aspects of trustworthiness. Achieving truthful or believable results is the aspect of trustworthiness called credibility (Ely et al., 1991; Erlandson et al., 1993; Lincoln & Guba, 1985; Miles & Huberman, 1994; Rodwell & Byers, 1997). Realizing the diversity in educational settings and individual students, I have provided sufficient information so that the reader can decide what results from the study are relevant in other situations. This is called transferability (Erlandson et al., 1993; Lincoln & Guba, 1985; Miles & Huberman, 1994; Rodwell & Byers, 1997). Although qualitative research is difficult if not impossible to replicate, the need for the results to be consistent and to explain the methodological decisions made in the course of the study ensures dependability (Erlandson et al., 1993; Lincoln & Guba, 1985; Miles & Huberman, 1994; Rodwell & Byers, 1997). Finally, by demonstrating that the researcher remained neutral and that the findings were based on the data collected during the study, confirmability can be achieved (Erlandson et al., 1993; Lincoln & Guba, 1985; Miles & Huberman, 1994; Rodwell & Byers, 1997).

Ensuring Trustworthiness During Data Collection and Analysis

I used triangulation, which is the use of multiple sources, methods and investigators during the collection and analysis of data, throughout this study. This ensures the trustworthiness of my findings (Creswell, 1998; Ely et al., 1991; Erlandson et al., 1993; Glesne, 1999; Lincoln & Guba, 1985; Mathison, 1988; Merriam, 1988; Miles & Huberman, 1994; Patton, 1980, 1990). According to Erlandson, et al. (1993, p. 137-138), "triangulation leads to credibility by using different or multiple sources of data (time, space, person), methods (observations, interviews, videotapes, photographs, documents) [and] investigators (single or multiple)." I collected data using multiple methods including classroom observations, documents of student work, and clinical interviews. By observing students, talking to them about their strategies for solving problems and copying students' work, between-method triangulation was employed. On the clinical interview protocols, I included the same problem types with different numbers to allow for within-method triangulation. Since this study was conducted over several months, I collected these different sources of data at various times from the thirteen participants. Finding patterns between data collected through different methods led to convergent validity (Ginsburg, 1997).

While developing this study, collecting data and conducting data analysis, I worked with a peer debriefing group. A peer debriefing group works as a support mechanism to both challenge and encourage each member throughout the research process (Creswell, 1998; Ely et al., 1991; Erlandson et al., 1993; Glesne, 1999; Lincoln & Guba, 1985; Spall, 1998). My peer debriefing group consisted of several doctoral students in education. We met on a regular basis to discuss the designs of our studies, emerging themes, and other aspects of conducting rigorous qualitative research. We offered suggestions and questions for each other's research projects and challenged each other to consider other possible methods for conducting our studies and interpreting the data. According to Lincoln and Guba (1985, cited in Spall, 1998), "Peer debriefing supports the credibility of the data in qualitative research and provides a means toward the establishment of the overall trustworthiness of the findings" (p. 280). Notes from each of our meetings serve as a paper trail of our discussions, decisions, struggles, and concerns.

I also address transferability, dependability, and confirmability through the details provided in the remaining sections. The selection of participants and the description of the setting provide information that will help the reader determine how this study may be applied to another setting. This relates to the transferability of my findings. By explaining the decisions I made throughout the process of data collection and analysis, I have provided support that further ensures the dependability of these findings. I have achieved another aspect of dependability through inter-rater reliability of coding of the data. By consistently making direct links to student work throughout the process of analysis and reporting of the findings, I have ensured the confirmability of my findings.

Researcher as Instrument

Recognizing the interaction among the researcher, participants and data collection and analysis in qualitative research, I wrote a statement entitled “Researcher as Instrument” before beginning this study (see Appendix A). Erlandson et al. (1993) points out that, “the researcher him- or herself becomes the most significant instrument for data collection and analysis” (p. 39). In writing my Researcher as Instrument statement, I had to consider my own experiences and beliefs regarding learning and teaching about fractions (Patton, 1990). In addition, I included reflections about what I expected to learn in the study, and I described what I was not comfortable discovering during this study. I have incorporated some of my relevant experiences and beliefs into the discussion of this study.

SAMPLE

Teacher Selection

To study students’ own approaches for solving problems and how they used physical models, the students had to be in a classroom with a teacher who supported each child’s unique development. I needed to find a teacher who used a problem-solving approach for teaching mathematics, believed in this philosophy, and utilized this approach for teaching fractions previously. Finally, the teacher, with the support of the school, had to be willing to participate in this study. Marla Bell[1] was recommended as a teacher who fit these criteria, and she enthusiastically volunteered to participate in the study.

Marla Bell strongly believed in the Cognitively Guided Instruction (CGI) philosophy and had taught using this approach for many years. She described herself as initially a very traditional teacher who taught every child the same procedures using the same materials. After receiving district training on a developmental approach for teaching reading, she wanted to use a similar approach for teaching mathematics and had an opportunity to attend her first CGI workshop. Initially, Ms. Bell worked alone in her classroom and tried to incorporate some of the CGI ideas in her teaching but found it very challenging. After attending another summer institute, Ms. Bell worked with a more experienced CGI teacher and fully implemented this philosophy in her mathematics class. During the following years, Ms. Bell continued to attend CGI professional development workshops, sometimes paying for the training with her personal funds. One of the workshops focused on teaching fractions using a CGI approach, so she started incorporating the unit into her class. In addition to being a CGI trainer, Ms. Bell took a leadership role in the mathematics professional development at her school.

Student Selection

Ms. Bell was in her third year teaching a third, fourth and fifth grade multi-aged class at a local public charter school. Since the teachers at the school

[1] To maintain confidentiality of the participants, pseudonyms are used throughout.

stay with the same class, she was teaching the fifth graders for the third year, the fourth graders for the second year, and the third graders for the first year. The fifteen students in Ms. Bell's class included: four fifth graders, five fourth graders, and six third graders. Two of the students received individual instruction from a special education aide and did not participate in mathematics lessons.

The remaining thirteen students participated fully in the study that included classroom observations and at least two interviews. I have included a list of students who participated, with their grades and ages at the beginning of the study, in Appendix B.

After observing the students during the first two to three weeks of the fraction unit, I identified a purposive sample of eight students to follow more closely over the course of the unit. A purposive sample incorporates a variety of students, including typical and atypical cases (Creswell, 1998; Erlandson et al., 1993; Glesne, 1999; Lincoln & Guba, 1985; Merriam, 1988; Miles & Huberman, 1994; Patton, 1980). Specifically I used maximum variation sampling, which included gender, grade level, and observed differences in ability or use of strategies. These eight students participated in an extra interview approximately half-way through instruction and I videotaped them more often than the other students during classroom interactions. Since I followed the students over numerous months, I started with enough students so that I would continue to have diversity in the sample to allow for potential attrition from the study. Through the use of maximally variable purposive sampling, others may transfer the common or reoccurring patterns in how students learned about fractions to other situations and future research studies (Erlandson et al., 1993; Lincoln & Guba, 1985; Miles & Huberman, 1994; Patton, 1980).

SETTING

This section describes the setting where students were learning about fractions. A rich, thick description provides detail about the setting so that readers can decide how the results of the study may transfer to their own situations (Creswell, 1998; Erlandson et al., 1993; Glesne, 1999; Lincoln & Guba, 1985; Merriam, 1988; Miles & Huberman, 1994; Patton, 1980). To understand the setting completely, it is useful to learn about the school itself and the underlying philosophy. I described both the classroom environment and the structure of instruction. Since the focus of this study is how students develop models of fractions, the classroom environment and instruction by the teacher provides the context in which this development occurs.

Overview of the School

The setting for this study was a public charter school with a mission statement that included ensuring *every* child meets his or her potential. To this end, the school utilized a developmental approach to teaching all subject areas and employed a multi-age level grouping of students where teachers stayed with the same students for multiple years. The focus of what students needed to learn was based not on their age or grade level, but where the child was academically. Limiting class sizes to fifteen students allowed the teacher to carefully monitor each child's progress and make adjustments to instruction throughout the year.

The school curriculum required teachers to use developmentally appropriate approaches to teaching mathematics and language arts. The teachers frequently taught other content areas using a thematic approach. Throughout the year, colleagues and consultant provided on-going professional development to support the teachers.

All of the teachers received professional development in Cognitively Guided Instruction (CGI), which provided the philosophical basis for teaching mathematics. Research on how children use their informal and intuitive knowledge to develop their understanding of mathematics concepts, especially in the area of number concepts, is the basis for CGI (Carpenter et al., 1999). Since teachers encourage students' individual mathematical development in a CGI classroom, this setting was ideal for examining children's approaches for solving problems and their use of physical models as the primary focus of this study. The underlying Cognitively Guided Instruction approach for teaching mathematics impacted the classroom organization, instruction, and the use of physical models.

Classroom Environment

Although Ms. Bell's rectangular classroom was smaller than many typical elementary classrooms, it was one of the larger ones at the school. In the front of the room, there were two white erase boards and enough space on the floor for the class to sit during whole-class discussions. Three tables were in the middle of the room – two rectangular and one circular – where students sat to do their work. Although students often chose to sit at the same place, they were not assigned seats and moved around as they worked on various assignments with different classmates. The back wall contained a shelving unit with a variety of materials related to instruction. Most of the mathematics manipulatives were located on shelves that were easily accessible for the students. There was also table near the back of the room where the special education aide worked with two students on a daily basis.

Mathematics Instruction

Ms. Bell's class structure was organized around her philosophy of teaching. As students entered the classroom in the morning, Ms. Bell expected them to work on a warm-up activity. Many of these activities required the students to write information in their math journals. After a brief class meeting, the class discussed the warm-up activity. The students spent the rest of the class time working on other mathematics activities. The amount of time spent on mathematics ranged from 45 to 90 minutes depending on the schedule for the day.

There was a typical cycle that occurred over one or two days. Ms. Bell gave students one or more problems to solve on small pieces of paper. They glued the problem in the mathematics journals and then solved the problem using one or more strategies. The teacher expected students to explain their strategy in detail so it was clear what they did and why. During this class time, Ms. Bell walked around the room and worked with individual students. In addition to writing observations about their strategies on a form, she asked questions to guide students in solving the problems. Often, Ms. Bell examined the journals after mathematics class and added notes to her observation form. Based on her observations, Ms. Bell selected several students to share their solution strategies during whole-class discussions. Each student explained and justified his or her solution to the class and answer questions often posed by classmates. Students appeared to pay attention because they asked questions – often ones they had heard Ms. Bell ask previously – and students "challenged" incorrect answers or statements that were not mathematically correct. Ms. Bell guided the classroom discussion by posing questions such as asking students to compare strategies or solve a different problem using the same strategy.

Due to the safe, positive classroom environment developed by Ms. Bell, students wanted to share their ideas and answers in front of the class. Many times all of the students wanted to share their answers so Ms. Bell employed various techniques. Sometimes she had students work in small groups to share their answers. Other times she had all students show their strategies on laptop white boards and then decide which strategies were similar to the ones on the main board. In this mathematics class, Ms. Bell challenged students to understand the mathematics content by engaging in problem-solving and classroom discussions.

Use of Physical Models

When students solved problems, they could make drawings or use any of the manipulatives available on the shelves in the back of the classroom. Students used drawings to figure out answers, as a second strategy to prove an answer, to represent what they did with manipulatives, or to justify an answer. Ms. Bell expected students to communicate how they solved problems by recording their actions and thoughts in their journals. The models that the teacher introduced through activities in class included two-colored counters, pattern blocks, and fraction strip kits. Other materials that students could access and utilize included colored tiles, base 10 blocks, and beans.

Two of the pre-fabricated manipulatives Ms. Bell introduced in class were the two-colored counters and the pattern blocks. The teacher introduced the two-colored counters as a set model. Following the steps and answering questions from an activity in the book *About Teaching Mathematics: A K-8 Resource* (Burns, 2000), students worked with twelve counters and divided the counters into thirds, fourths, and sixths. Students named equivalent fractions, such as 1/3 equals 2/6 and 4/12, as they sorted the counters into different groups. Later in the unit, Ms. Bell asked the students to look for relationships between the yellow, red, blue and green pattern blocks. After students made their lists, the teacher divided the class into three smaller groups to share. Then she asked students to think about if each shape was equal to one whole, what was the value of the other shapes? For example, if the trapezoid was one whole, the triangle was 1/3. The teacher gave students time to solve the problem before coming together to discuss their answers.

The other primary manipulative students used in class instruction were the fraction kits. They used 3” by 18” construction paper strips to construct their fraction kits early in the unit. Ms. Bell found the directions for making the kits and the activities to use with the kits in *About Teaching Mathematics: A K-8 Resource* by Marilyn Burns (2000). Each student in the class made a fraction kit with a whole, halves, fourths, eighths, and sixteenths. During this lesson, the discussion focused on making equal-sized pieces and using relationships such as folding 1/4 in half to make two 1/8 pieces. Later in the unit, the students added thirds, sixths, and twelfths to their kits.

After the students completed the fraction kits, Ms. Bell introduced the games “Cover-up” and “Uncover.” The objective of the first game was to cover-up the whole piece with fractional pieces after rolling a die with 1/2, 1/4, 1/8, 1/8, 1/16 and 1/16 on each side. The teacher showed students how to record their rolls in their journal and how to find the sum of their pieces. If a student won the game, they had to show their answer was equal to one. On the other hand, a student that lost the game had an answer less than one and had to say how much more was needed to make a whole. After noticing how students chose to add the fractions, Ms. Bell posed a question about how they decided which numerator to use. The discussion revolved around using sixteenths because all of the fractions in this game could be converted to sixteenths or using the largest possible fraction piece by combining smaller fractions when possible. The objective of “Uncover” was to start with two-halves and remove pieces to make zero. After rolling the die, the students either had to remove a piece, make an equivalent trade, or pass. Unlike the previous game, Ms. Bell did not require students to write anything down in their journals but expected them to justify their trades verbally to their partners.

Ms. Bell gave students some warm-up activities and homework assignments where the fraction kit could be used to answer the questions. Otherwise, she did not require students to use the fraction kits and they chose when and how to use them. For example, when students solved an equal sharing problem and ended up with two fractions in the answer, some children used the fraction strips to find the sum. On occasion, the teacher specifically suggested the students use the fraction strips to help solve a problem.

DATA GENERATION/COLLECTION

Before the fraction unit, I observed informally in the classroom. I conducted clinical interviews prior to and after instruction with all 13 students and with eight of the students approximately in the middle of the unit. I videotaped and summarized all of the clinical interviews. Throughout the fraction unit, I videotaped the mathematics class on a daily basis. I collected copies of all student work, and Ms. Bell provided some of her observation notes.

Initial Entry into Classroom

I began visiting Ms. Bell's class approximately four weeks before the start of the fraction unit. The teacher told the students about the study and introduced me when I started visiting the classroom. Initially, I was in the classroom to get to know the students' names and the classroom routines established prior in the year. This also gave the students an opportunity to become comfortable with my presence in the classroom. Since I entered the classroom before data collection and remained for several months, I met the criteria for prolonged engagement. Prolonged engagement requires remaining in the setting for an extended amount of time and is an important step in establishing credibility for a study (Creswell, 1998; Ely et al., 1991; Erlandson et al., 1993; Glesne, 1999; Lincoln & Guba, 1985; Merriam, 1988; Patton, 1990). During the entry phase, I visited the classroom three to four times per week.

Prior to data collection in the classroom, the teacher sent a letter home to each student's parents or guardians explaining the research project. This letter included a consent form requesting permission to videotape and audiotape students during the mathematics class and for individual clinical interviews as well as to copy students' class work on fractions (see Appendix C). The letter also assured parents that specific information about their child's learning remained between the teacher, the parents and the researcher. When I shared the results of this study publicly, students' names and other identifying information would remain confidential. All participants chose their pseudonyms for the reporting of the data.

I offered to help the Ms. Bell during mathematics class during this initial entry period when I was not collecting data. In addition to wanting to be useful to her since she was being so helpful in conducting this research, I wanted an opportunity to get to know the students and become familiar with the classroom environment. For example, I listened to children solving different problems and asked them guiding questions. The teacher and I talked about what the students did during recess, lunch or after school. In addition, I made notes about the students from my observations and provided them to the teacher.

Clinical Interviews

Development of Interviews

While becoming acclimated to the classroom, I developed the questions for the clinical pre-interview (see Appendix D). The purpose of the pre-interview was to establish a baseline about each child's initial approaches to solving problems with fractions and understandings prior to instruction. I based many of the questions on equal sharing situations that the teacher intended to use in class for the unit. I included questions specifically related to order and equivalence of fractions both in a problem solving context and in symbolic form. After Ms. Bell reviewed the pre-interview protocol, she indicated that some of the problems were relatively simple for her advanced students so I added computation word problems, including some based on test questions written for middle school students, as challenge problems for a few students.

Throughout the development and implementation of the clinical interviews, I took certain steps to ensure the quality of the interviews and data collected. Initially I developed a detailed interview protocol which included the specific questions, probes and prompts related to each question, and decision points about when to give a more challenging or an easier question (Ginsburg, 1997; Goldin, 2000). Students had time to work on the problem independently without questions before being prompted or probed about how they thought about the problem (Goldin, 2000). I carefully selected and worded each question so that the tasks were directly related to students' understanding of fractions; this assured content validity of the interview (Ginsburg, 1997). Within each group of similar problems, I presented easier questions that most students should be able to answer in the beginning and more difficult questions that fewer students could answer near of each subset of similar questions (Goldin, 2000). The last question for the interview was always an easier problem that I expected most students could solve correctly.

The purpose of developing a structured interview is to make the interview process replicable; that is, the interview could be repeated with other students in a similar fashion (Ginsburg, 1997; Goldin, 2000). After I developed the pre-interview protocol, I conducted the interview with four upper elementary students who were not participants in the study (Ginsburg, 1997; Goldin, 2000). By developing a detailed interview and piloting it with students, I was more consistent when conducting the interviews with the study students (Goldin, 2000). During the pilot interviews, I examined whether the questions required the student to use or explore ideas about equivalence and order of fractions. This provided evidence of construct validity in that the questions "provide direct evidence concerning the thought processes" under investigation (Ginsburg, 1997, p. 178). I made adjustments based on these pilot interviews.

Conducting Interviews

One week before the teacher began the fractions unit, I conducted clinical pre-interviews with all thirteen students. During a clinical interview, the researcher asks a child specific questions and probes to understand how the student constructed this answer (Ginsburg, 1997; Goldin, 2000). Often this requires asking, "Why?" or "How did you figure that out?" Ginsburg (1997) explains that, "The clinical interview can help you understand how children construct their personal worlds, how they think, how their cognitive processes (at least some of them) operate, how their minds function" (p. 28). Based on the student's response, the researcher starts to hypothesize about the student's understanding and can decide to probe further, ask a previously written question or develop a new question. Goldin (2000) explains that an interview "offers the opportunity for research-based inferences about students achievement of higher and deeper mathematical understandings" (p. 524).

I conducted the clinical interviews during or after school as arranged with the teacher, principal and parent of each student. Each videotaped interview lasted approximately forty to sixty minutes, took place in a quiet place at the school such as an empty classroom or the library. When I was unable to complete an interview in one session, I continued the interview on a subsequent day. When a student used paper to solve any problems, I wrote the student's name, date, and question number on the sheet of paper and kept the paper as a document for part of my analysis. Students had the same manipulatives available during the interviews as they had available in their classroom. These manipulatives included items such as pattern blocks, base-ten blocks, counters, rulers, and a fraction kit. On some occasions, I asked students to explain why they chose to use a specific method or manipulative to solve the problem, such as drawing a picture or using one of the manipulatives.

During the interviews, I used "member checking" to assure that I understood how he or she solved the problem. As a student in college mathematics courses, I realized that it was often difficult for me as an adult to recall a process of how I solved a problem in the past; therefore, I member checked how students solved specific problems while the information was still fresh in their minds. This first level of member checking involved restating how the child solved the problem and asking for clarification so that I could understand the specific thought process the child used to solve problems. Member checking was one method of assuring that I was capturing the students' thinking as closely as possible.

I conducted additional clinical interviews in a similar manner later during the school year. Eight students participated in a mid-interview (see Appendix E) and all thirteen students completed the post-interview (see Appendix F) after the conclusion of the unit. The questions were similar to those on the pre-interview questions. I made some adjustments in the later interviews because the original pre-interview questions did not provide useful data.

Summarizing Interviews

Initially I summarized each child's response to each question during the clinical interviews in a relational database. Each record in the database was one student solving a single problem. Using the videotape from the clinical interview, I described how a student solved a problem including: how the child used manipulatives, how the child drew pictures, and the child's verbal explanations. When I included notes about my interpretations of how the student solved a problem, I clearly indicated that they were my interpretations and not part of the description. To assure the plausibility (or credibility) of my interpretations, I used multiple examples of observed behavior to support how and why I made specific inferences (Ginsburg, 1997). There were also fields in the database to write the student's answer for each question and check boxes for the correct answer and viable strategy.

Classroom Observations During the Unit

In addition to conducting clinical interviews with the students, I was a participant observer in the classroom on a daily basis while the teacher implemented the fraction unit. When the teacher led a whole group activity, my primary role was as an observer. I video taped the discussions, and used these videotapes to summarize the events in the classroom. On occasion, I transcribed discussions or specific comments verbatim. I used these video tapes and resulting summaries to keep a record of the lessons presented to the whole class, as well as serve as data about the students' participation in these discussions.

When students were working individually or in small groups, my role was more of an observer participant. As students worked, I walked around the classroom to videotape what individual students were doing, especially focusing on the eight students that I identified early in the unit. At times I just watched and listened to students who worked on an assignment. When students explained their answers to classmates or talked about how to solve a problem, I wanted to observe these interactions. At other times, especially when students were not talking about how to solve the problems, I asked them questions about their work. Sometimes I videotaped Ms. Bell working with an individual or small group of students. Again, this information was member checked with the students during the individual or small group conversations. Using a combination of the videotape and copies of student work, I summarized these interactions. In some instances, I transcribed the interactions with the students when I believed the discussion provided useful insight into how a specific student was solving problems.

Collecting as much data as possible about how students learned and understood fractions during the unit provided opportunities to connect their apparent development on the interviews to activities or discussions during instruction.

I collected a variety of artifacts from the classroom observations, including copies of student work and observation notes provided by the teacher. Throughout the unit, Ms. Bell made notes about individual students as she observed them and read through their completed assignments. She used these notes to identify students to share their solutions during class discussions, to guide her in addressing questions with specific students and as resources for talking with parents at conferences. Ms. Bell shared many of her observations during the on-going interviews and also provided some copies of her notes. This data – teacher interpretations about what students were learning and understanding – provided another viewpoint about the children's learning of fraction concepts over time.

Since I was in the classroom several weeks before the fraction unit started until I finished with post-interviews, I developed a trusting relationship with the teacher and students. Persistent observation provided a depth of understanding about the classroom culture as well as the individual students (Creswell, 1998; Ely et al., 1991; Erlandson et al., 1993; Glesne, 1999; Lincoln & Guba, 1985). It also allowed me to identify aspects classroom learning environment that were relevant to understanding how students were learning fractions.

DATA ANALYSIS

My primary data sources were clinical interviews, videotapes from the classroom and copies of all student work generated over the course of the unit. I identified specific problems from the student interviews that provided the most useful data about students' approaches for solving order and equivalency problems. I coded children's solutions and conducted an analysis of the patterns. After identifying the emerging themes, I reviewed the classroom data to look for specific and relevant connections.

Clinical Interviews

I initially focused on the clinical interviews for the analysis. These interviews provided data about each child's specific approach to solving similar problems over time. After summarizing interviews for all of the students, I selected specific problems for the multiple iterations of coding.

Problem Selection

After I finished with transcribing the clinical interviews, I focused the analysis on the questions most directly related to order and equivalency. I reviewed the pre-, mid-, and post-interviews and selected the questions that provided the most information about comparing and ordering fractions. All of these problems given during the clinical interviews are listed in Tables 4b, 4c, and 4d. The first questions provided a scenario where students in the problem had solved an equal sharing problem and obtained multiple answers. I provided equivalent amounts in all of the answers, but the students had to determine whether the fractions were equal and justify their answers. These problems provided a context for the fractions. In contrast, the remaining ordering and comparing problems were in symbolic form devoid of a context. Students had to order three or four fractions in the second set of questions. Students had to compare two fractions and determine if they were the same or different in the third set of problems. If students said they were different, I asked them which one was larger or why. For all of these problems, I asked students to justify their answers.

I included the easiest numbers prior to instruction in the pre-interview (see Table 2). The first context equivalence problem used one-half, a fractional amount that is usually understood early on by children. The second context problem used a non-unit fraction (2/3), and I only presented it to students who successfully solved the previous problem. I asked students to order three unit fractions, which also tends to be an easier task. Finally, the last problems asked students to compare two fractional amounts where the amounts are unequal for one problem (2/3 and 3/4) and equivalent for another (2/12 and 1/6).

Table 2: Pre-interview Order and Equivalence Problems

Pre-interview
The children in Ms. Jones' class were solving an equal sharing problem, where all the cakes were the same size. Some got one-half of a cake for their answer. Others got two-fourths of a cake for their answer. Are these the same amounts of cake or different amounts of cake? One student said the answer could also be three-sixths of a cake. Is this student correct? Can you think of another answer that would be correct?
Another day the children in Ms. Jones' class were solving a different equal sharing problem, where all the cakes were the same size. This time some got two-thirds of a cake for their answer. Others got six-ninths of a cake for their answer. Are these the same amounts of cake or different amounts of cake? One student said the answer could also be eight-twelfths of a cake. Is this student correct? Can you think of another answer that would be correct?
Arrange these fractions in order from smallest to largest: One-fourth, one-fifth, and one-third? How do you know ___ is the smallest? How do you know ___ is the largest?
Are these fractions different amounts or the same amounts: Two-thirds and three-fourths. How do you know? Which one is larger? How do you know?
Are these fractions different amounts or the same amount: Two-twelfths and one-sixth? How do you know?

I purposely included more challenging numbers on the mid-interview (see Table 3). Since students had been working with fraction strips, I used specific numbers related to the fraction strips. At the same time, I chose some of the numbers to determine how the children used relationships for fractions they were not as familiar with. For example, children had worked with eighths but not twelfths, so they could not rely on the fraction strips to compare 2/8 and 3/12. They also had not worked with sixths, so I expected the problem comparing 5/6 and 10/12 would be more difficult. If students decided that these fractions were equivalent, I asked if they could think of a third fraction that could also be equal to 5/6 and 10/12. Students had to order three fractions that had the same numerator, which was slightly more difficult than the similar pre-interview problem with only unit fractions. I made the last problem easier so that students finished the interview feeling successful. It was the only comparison problem on the mid-interview with fractions included in the fraction kits.

Table 3: Mid-interview Order and Equivalence Problems

Mid-Interview
The children in Ms. Lee's class were solving an equal sharing problem, where all the cakes were the same size. Some got two-eighths of a cake for their answer. Others got three-twelfths of a cake for their answer. Are these the same amounts of cake or different amounts of cake? What's another way to say that amount of cake with fractions?
Another day the children in Ms. Lee's class were solving a different equal sharing problem, where all the cakes were the same size. Some students said the answer was ten-twelfths of a cake. Others said five-sixths of a cake for their answer. Are these the same amounts of cake or different amounts of cake? What's another way to say that amount of cake with fractions?
Arrange these fractions in order from smallest to largest: Four-sixths, four-twelfths, and four-eighths. How do you know ___ is the smallest? How do you know ___ is the largest?
Are these fractions different amounts or the same amounts: Three-fourths and four-thirds. How do you know? Which one is larger? How do you know?
Are these fractions different amounts or the same amounts: Two-thirds and four-sixths. How do you know?
Are these fractions different amounts or the same amount: three-sixteenths and five-eighths? How do you know? Which one is larger? How do you know?

I carefully selected the numbers on the post-interview to examine students' abilities to make models for fractions they had not worked with as frequently (see Table 4). I only included one context problem on the post-interview. I chose fifths and tenths because these were realistic fractional amounts not included in the students' fraction strips. By this point many students used a "double the numerator and double the denominator" strategy for finding equivalent fractions. I included the other possible answers because they could not be found using this strategy. The problem ordering one-half, one-sixth, two-thirds and three-fourths was based on a Texas Assessment of Academic Skills (TAAS) test from Spring 1996 (Grade 5, problem #19).

Table 4: Post-interview Order and Equivalence Problems

Post-interview
The children in Ms. Lee's class were solving an equal sharing problem, where all the pancakes were the same size. Some got two-fifths of a pancake for their answer. Others got four-tenths of a pancake for their answer. Are these the same amounts of pancake or different amounts of pancake? What's another way to say that amount of pancake with fractions? Could six-fifteenths (or 10/25, 14/35) be the same amount as two-fifth and four-tenths?
Arrange these fractions in order from smallest to largest: two-thirds, three-fourths, one-sixth, and one-half? How do you know ___ is the smallest? How do you know ___ is the largest?
Arrange these fractions in order from smallest to largest: seven-fifteenths, two-sevenths, and six-tenths? How do you know ___ is the smallest? How do you know ___ is the largest?
Are these fractions different amounts or the same amounts: four-sixths and three-fifths. How do you know? Which one is larger? How do you know?
Are these fractions different amounts or the same amounts: two-eighths and three-twelfths. How do you know?

The students in Ms. Bell's class had not learned how to compare these fractions using common denominators so I expected them to use a variety of strategies. I asked students to order the fractions 7/15, 2/7, 6/10 because they did not have fraction kit pieces in these denominators, and it was difficult to convert these fractions to a common denominator. Students had to consider relationships to compare four-sixths and three-fifths, and they could not solve it using their fraction strips. Although students could compare 2/8 and 3/12 on the last problem using their fraction strip pieces, I asked for a second strategy if they used the fraction strips.

Coding of Data

I coded the interview data using a recursive process of identifying, describing, reviewing, and revising the emerging categories multiple times during data analysis. I sorted the data, identified possible emergent categories, described these categories in greater detail, resorted the data, and further refined the categories. I shared this information with my peer debriefing group or my advisor. When the coding of the data did not adequately describe the richness or the variety of approaches utilized by students, I tried a different process to analyze the data using a combination of coding in a database and by hand.

I originally organized and coded the data using fields in the database, but this did not allow me the flexibility needed to identify emerging categories. Therefore, I transferred the key information about each child's solution to an index card, which I then organized multiple times. Using the constant comparative method described by Glaser and Strauss (cited in (Erlandson et al., 1993), I sorted the cards based on the primary way that students were focusing on the fractions in the problems. As I found different students who had similar approaches and reasoning, the categories began to emerge from the data. I named the categories, described them, and provided examples of student work on tables to further clarify my understanding of each category. I shared these tables with my peer debriefing groups and advisor for further scrutiny and refinement. After making multiple revisions to these categories, I compared these categories to perspectives described by Smith (1990; 1995) and further refined them. I identified the following perspectives: Limited, Pieces, Part-Whole, Unit Fraction, Within-Fraction, Between-Fraction, Equivalence, and Transform. I describe these eight perspectives in detail in Chapter 4.

With assistance, I modified the database so that I could code the interview questions related to order and equivalence by the perspective or perspectives. I also connected additional information to each perspective: a brief summary of what the child did, what physical models the child used, and how the child used the physical model. Usually I started with one question and coded the data for that particular question for all of the students. This helped with consistency in coding similar approaches by different students.

I only coded a perspective once for a single problem even if a child used it multiple times. If there were multiple perspectives within a single problem, I coded each one. When it was not apparent what students were thinking, such as when they were moving around manipulatives or said they could not think of a way to prove or explain their answer, I did not code the data.

One of the challenges was clarifying when to use some of the perspectives. I had to determine when to use the equivalent perspective for recall facts. When the student said fractions were equal and proceeded to refer to an easily observable pattern such as "the numerator and denominator are doubled," it was assumed that the student used this pattern to determine the fractions were equal and I did not code it as the equivalent perspective. I only coded the perspective related to the observed pattern. When students used a strategy that was not observable by looking at the numbers, for example proving that the fractions were equal using a drawing or a transformation procedure, I coded it both ways: as the equivalent perspective and the perspective used to justify the answer. To differentiate the between-fraction perspective from the transform perspective, I looked at whether students acted upon the numerator and denominator as individual components or used a procedure for generating equivalent fractions. Students who used a transform perspective were multiplying or dividing one of the fractions from the problem by a fraction equal to one. When students had a between-fraction perspective, they often worked with one component of the fraction, such as the numerator, separately from the other component. Instead of using fractions in their work, they multiplied or divided a whole number times by another whole number. I referred back to the students' written work when I had questions about one of these perspectives.

Inter-rater Reliability

After I finished coding the data, a mathematics education colleague and I established inter-rater reliability through a process of coding, discussing and recoding. Using a table summarizing the eight perspectives similar to the one provided in Chapter 4 and narrative descriptions of each of the perspectives, my colleague coded the data for four students. I sent her the date with the specific problems selected for the analysis, children's answers and the summaries of the interview. After she finished coding the data, we compared how we had coded the data for these four students. We went over each problem where we had any disagreements and discussed the child's solution until we agreed about the coding.

Through this process, we clarified how to differentiate between different perspectives and how much detail to include in our coding. We both coded two additional students independently based on our shared understanding of the perspectives. The inter-rater reliability for this second set of students was 70%. It was calculated by dividing the number of agreements by the total number of agreements, disagreements, and extra codes (where one of us coded data that the other person did not code). Again, we went over all of the children's solutions where we disagreed or coded different information until we reached a consensus.

Analysis of Patterns

After coding all of the order and equivalence problems for all of the students, I began to look for patterns by slicing the data in multiple ways. Using the database, I sorted the data by problem, by perspective, and by student. I copied these different reports into a spreadsheet so that I could examine the patterns. By focusing on each problem, I could hypothesize whether the particular numbers influenced what perspective students used. By focusing on a particular student, I could determine whether a student used the same perspective to solve multiple problems. These patterns could be evidence of internal consistency in the perspective of an individual student (Ginsburg, 1997). In addition to making sure that I was being consistent in my coding of data, slicing the data by perspective allowed me to see patterns in which students or which kind of problems students solved using the same perspective. Since I connected the use of physical models to this coding, I looked for patterns between the use of physical models, problems, perspectives and students. I used these reports to analyze the data as well as search for and verify plausible themes.

Connecting Classroom Observations to Clinical Interviews

The final step in data analysis was connecting the clinical interviews to learning in the classroom. Ms. Bell asked a variety of problems related to comparing and ordering fractions, primarily after the mid-interview. Initially the students compared fractions using the manipulatives in the classroom. After the mid-interview, the teacher presented activities related to using benchmarks. Many of these assignments were warm-up activities that led to discussions about comparing and ordering fractions.

Table 5 lists the order and equivalence problems Ms. Bell gave during the first half of the fraction unit. During the early part of the unit, Ms. Bell usually asked students to compare fractional amounts with the manipulatives that were being used in class. For example, the students had made fraction strips and were playing fractional games during this time so she asked about equivalent relationships in the fraction strips. She also presented an activity with pattern blocks where students looked for equivalent relationships between the pattern block pieces. The class made a chart related to the length of their names. Then the teacher asked students to write statements about their observations from the charts. Some students wrote about relationships that compared fractional amounts in the chart.

Table 5: Order and Equivalence Class Problems Prior to Mid-interview

Class Work (before the mid-interview)	
March 6 (Hmwk)	List all combinations of fractions from your fraction kit that are equivalent to the fraction named. (List only halves, fourths, eighths, or sixteenths.) 1/4 and 1/1
March 7 (Hmwk)	List all combinations of fractions from your fraction kit that are equivalent to the fraction named. (List only halves, fourths, eighths, or sixteenths.) 1/8, 1/4, 1/2, 1/1
April 9	Comparing data from a chart (related to the number of letters in names)
April 11	Comparing pattern block pieces

During the second half of the unit, the teacher posed multiple questions related to comparing and ordering fractions and largely focused on using benchmarks. Ms. Bell often gave these questions, listed in Table 6, as warm-up activities that students worked as they came into class in the morning. Then the entire class gathered for a discussion about the problems before proceeding to the day's primary math assignment. Near the end of the unit, Ms. Bell gave the students two equal sharing equivalence questions.

Table 6: Order and Equivalence Class Problems After Mid-interview

Class Work (after the mid-interview)	
April 17 (Warm-up Activity)	Classify these fractions as nearest to 0, ½ or 1. Explain each decision 12/10 15/29 6/5 49 8/9 2/7 75/80 7/15 1/20 53/100 9/10 2/12
April 19 (Warm-up Activity)	Classify the following fractions as closer to 0, 1/2, or 1. Make a chart. Be able to explain. Order the fractions from least to greatest. 5/9 2/5 2/12 8/9 5/4 2/10 5/8 1/8 4/3 2/6 8/7 14/12
April 23 (Warm-up Activity)	Which of the following is/are equal to 3/4? Prove. 4/6 6/8 9/12 12/16

Class Work (after the mid-interview)	
April 26 (Warm-up Activity)	Name a fraction that is close to 1, but not more than 1. Name another fraction that is even closer to 1 than that. Explain (in writing) why you believe the fraction is even closer to one than the previous fraction.
April 30 (Warm-up Activity)	Compare 3/8 and 4/7. Are they equivalent? Is one larger? Which one? How do you know? Explain in writing.
May 4	Who gets more clay: A child at a table where 3 children are sharing 2 boxes of clay equally or a child at a table where 6 children are sharing 4 boxes of clay equally?
May 4	Who gets more clay: A child at a table where 4 children are sharing 3 boxes of clay equally or a child at a table where 12 children are sharing 8 boxes of clay equally?

Using the analyses of the clinical interviews, I looked for specific assignments related to comparing and ordering fractions from the classroom interactions that might demonstrate why or how a specific pattern emerged for an individual student. Ginsburg (1997) suggested this process of comparing what occurred in the interview with data collected through other methods leads to convergent validity. For example, if a student used a new strategy on the mid- or post-interview, could I identify a specific instance within class where the student learned about this strategy? If a student chose to use a specific manipulative during the interview, could I find specific instances of that child using that manipulative during mathematics class? If a child struggled with certain types of problems during an earlier interview, could I find examples of how the student was developing an understanding of the concept during class? Relating the classroom and interviewing data helped “obtain deeper insight into children’s thinking than can any one method alone” (Ginsburg, 1997, p. 180).

SUMMARY

The focus of this study is how children solved problems involving comparing and ordering fractions and the role of physical models in their solutions. I conducted this research in a combination third, fourth, and fifth grade class where the teacher utilized a CGI approach for teaching mathematics. To record the classroom observations of the fraction unit, I videotaped every lesson and made copies of all student work. I conducted clinical interviews with all thirteen students in the class twice – prior to instruction and immediately following the conclusion of the unit. I conducted an additional interview in the middle of the unit with eight of the students. To make each interview consistent, I developed a detailed protocol with problems, specific follow-up questions, and decision points. The teacher provided copies of some of her observational notes. To analyze the data, I identified and coded specific problems from the clinical interviews. I used the classroom observations to connect results from the interviews to instruction. Throughout the process of data collection and data analysis, I have taken steps to ensure the quality of the findings.

The next two chapters focus on the finding of the study. Chapter 4 describes the perspectives that emerged from the coding of the data. After an overview of the perspectives, each one is illustrated in detail and supported with specific examples from student work. Chapter 5 describes the themes that emerged from the analysis of patterns in the data.

Chapter 4: Perspectives

The purpose of this study was to examine the relationships between the use of physical models and students' developing understanding of order and equivalency with fractions. Students from a third, fourth, and fifth multi-grade class participated in this study over several months as they were learning about fractions. In this Cognitively Guided Instruction class, the teacher gave students a variety of mathematics problems to solve and always expected them to use mathematical reasoning to explain and justify their answers. Although the teacher introduced a variety of physical models and planned some structured activities with the physical models over the course of the unit, students chose how or if they wanted to use physical models, including drawings, for most of the problem solving activities.

In addition to being observed during the mathematics class, all of the students participated in individual clinical interviews prior to instruction and after the conclusion of the unit. Several students also participated in interviews approximately midway through instruction. Even though physical models were available during the interviews, I did not expect or require students to use any of the materials. I summarized or transcribe all of the data from the interviews. I analyzed specific problems that focused on comparing and ordering fractions both in terms of how students approached these problems and used the physical models. Through this process, the perspectives emerged as ways that students thought about fractions to make comparisons about relative size or equivalence. I describe these perspectives in this chapter

OVERVIEW OF PERSPECTIVES

Perspectives are based on the types of relationships that students attended to when solving comparing and ordering fraction problems. These perspectives, which are summarized in Table 7, emerged through the repeated sorting of interview data. They are organized from least sophisticated to most sophisticated to some degree. Important relationships that describe the ways that students thought about the fractions as they made judgments about relative size are identified for each perspective. Within each perspective, there were different levels of complexity in students' approaches to solving problems. This provides some information about the variability within a single perspective.

These perspectives are different from strategies in that they extend beyond what students did to solve problems to identify the fraction relationships students were aware of and addressed in solving the problem. For example, a student with a between-fraction perspective focused on the relationships across numerators and across denominators; however, this does not indicate whether the student used additive or multiplicative strategies to solve problems.

Table 7: Summary of Perspectives

Perspectives	Description of the perspective	Key Relationships	Levels of complexity
Limited Perspective	Student does not have an understanding of fractions that allows him/her to answer questions about comparing and ordering fractions.		
Pieces Perspective	Focus on fractions as pieces independent of the whole	Size of fraction is seen as an absolute amount, often based a specific manipulative Relationship of the piece to the whole is not mentioned/apparent Some equivalent relationships between pieces may be included	1) Pieces of a certain size or shape represent specific fractions 2) Recreate fraction by drawing similar to the manipulative 3) Maintain relationships between different-sized pieces
Part-Whole Perspective	Focus on fractions as parts of a whole	Relationship to the whole is always apparent Denominator tells how many parts to divide the whole Initial relationships are derived by repeated halving Use recall facts to make equal-sized pieces that maintain fractional relationships	1) Divide whole into the correct number of pieces 2) Try to make equal-sized pieces 3) Use facts to create equal-sized pieces
Unit Fraction Perspective	Focus on unit fractions	Unit fraction is based on how many pieces in a whole Iterate the unit fraction to make a composite fraction	1) Only compare unit fractions 2) Extend to non-unit fractions 3) Recognize when the unit fractions do not provide sufficient information to compare/order fractions

Perspectives	Description of the perspective	Key Relationships	Levels of complexity
Within-Fraction Perspective	Focus on the relationship between the numerator and denominator $\frac{N}{D}$	Numerator is half of the denominator, equals 1/2 Numerator and denominator are equal, equals 1 whole Approximate relationship to compare to 0, 1/2, 1	1) Exact relationship between numerator and denominator 2) Approximate relationship between numerator and denominator
Between-Fraction Perspective	Focus on the relationship between numerators and/or denominators $\frac{N}{D} \rightarrow \frac{N}{D}$	Identification of relationship between like terms (i.e. numerators) Relationships maybe additive or multiplicative	1) Double/halve numerator and denominator 2) Maintain ratio relationship
Equivalence Perspective	Focus on the relationship between equivalent fractions	Relationships between fractions in the physical materials Relationships derived by extending beyond the physical materials	1) "Recall facts" based on relationships from physical models 2) Relationship between unit fraction and dividing it into two pieces 3) Extend relationships based on recall or generated facts
Transform Perspective	Focus on using rules to compare and order fractions	Use of other perspectives to explain why transformation rules	1) Multiply/divide by n/n to compare or generate equivalent fractions 2) Convert to common denominator to compare fractions

Even though I generally ordered the perspectives in Table 7 from least to most sophisticated, there are exceptions. The limited perspective consisted of approaches that were based on an inadequate understanding of fractions, so this was definitely the least sophisticated. When students had a pieces perspective, they were more successful with solving problems; however, they were not making connections to the whole which is an important feature of fractions. Students with a part-whole perspective were aware of the relationship between the part and the whole. The unit fraction perspective required a higher level of understanding because students were building from the part-whole perspective and had to consider the relative size of fractions without relying on physical representations. The arrangement of the within-fraction, between-fraction, and equivalence perspectives may not accurately portray the relationships between these perspectives. It might be argued that all three of them are equally sophisticated and the differences may be a result of specific numbers in the problems they are solving. Students' ability to choose the most appropriate perspective and move flexibly between these three perspectives is an indication of a strong conceptual understanding of fractions. The transform perspective is listed last because it describes the way most students are taught how to compare and order fractions. Even though students may be able to use the transform perspective to solve problems, it does not mean their understanding of fraction concepts related to comparing and ordering fractions is more advanced than students who chose to use other perspectives.

Each of these perspectives is described in greater detail for the remainder of this chapter. To illustrate each of these perspectives, I have integrated examples primarily from the student interviews throughout. The key relationships and the levels of complexity outlined in Table 7 are further elaborated through the children's solutions. To give an indication of the importance of each perspective in students' approaches for solving problems, the number of students that used each perspective is included within these descriptions. For a quantitative summary of the frequency of each of these perspectives in all of the interviews, refer to Appendix G.

LIMITED PERSPECTIVE

When students had a limited perspective about fractions, they were not successful in solving order and equivalence problems. Often, students based their answers and explanations on faulty reasoning. Even though some of the answers were correct, students did not have the ability to explain and justify their answer using appropriate mathematical reasoning. The lack of understanding about fractions prevented these students from comparing and ordering specific fractions on specific problems during the pre-interview. There were several reasons for these difficulties. Some of these students did not seem to have a physical or mental representation that they could connect with the fraction terms. Other students relied on whole number relationships, most likely based on their previous experience with comparing and ordering whole numbers. Finally, some students made up rules to manipulate the numerators and denominators. These various approaches indicated that some students were not able to identify and use relationships for fractions to help them solve problems.

Certain students were unable to connect physical or mental representation with the fractional amounts. For example, on the pre-interview, one student stated that 1/2 and 2/4 were equal to each other because his dad told him this was true, but he did not provide proof or extend his answer to include another fraction such as 3/6. Another student attempted to represent 3/6 using cubes by placing a group of six cubes below a group of three cubes. Her lack of understanding of 3/6 prevented her from representing the fraction.

Students sometimes relied on their previous experience with whole numbers to compare fractions. Fractions in certain problems permitted students to correctly answer the question using this invalid strategy, even though it did not address the fractional relationship between the numerator and denominator. For example 3/4 is larger than 2/3 because 3/4 has larger numbers. Often these initial strategies led to incorrect conclusions. When asked to order 1/4, 1/5 and 1/3 from smallest to largest, some students used whole number relationships to explain that 1/3 was smallest because 3 was the lowest number and 1/5 was the largest because 5 was highest number.

Other students wanted to use a rule to help them determine the answer, but they made up their own rule to compare and order fractions. Christina decided that she could find other equivalent fractions by adding 1 to both the numerator and denominator. She claimed that another solution for the problem with 8/12 was 9/13. She also used this procedure to explain why 2/3 and 3/4 were equal to each other. Joey used a rule that if two unit fractions are added together, the result is the next higher unit fraction. He used this rule to explain that 2/12 and 1/6 were not equal to each other because 1/12 plus 1/12 equaled 1/11. When asked to compare 2/3 and 3/4, Todd decided that these fractions were equal to each other because he added both the numerator and denominators to get 5/7 and then flipped the fractions to make 3/2 and 4/3 and added them together to get 7/5.

Although 7 of the 13 students had a limited perspective for solving at least one problem on the pre-interview, all of the students had other perspectives that helped them solve other order and equivalence problems on subsequent interviews. Students rarely used this perspective on subsequent interviews. The following perspectives indicated that students had some basic understandings of fraction concepts that allowed them to solve more problems correctly.

PIECES PERSPECTIVE

Students who focused on specific fractions as a certain shape and size had a pieces perspective. The primary relationships were based on the fraction as pieces, and students did not make connections between the pieces and the wholes. The pieces perspective was manifested in both students' use of the physical materials and their drawings. Certain physical materials, including the pattern blocks and fraction strips, allowed students to represent and act on the materials without explicit connections to the size of the whole.

When students connected the pieces perspective with a particular manipulative, they focused on the size or shape as an absolute amount (or shape) and not in relation to the whole. For example, a student who pointed at pieces of pattern blocks and named a trapezoid 1/2 and a triangle 1/6 used a pieces perspective. At times, students identified equivalent relationships between the pieces. A student explained that 2/6 equaled 1/3 because two triangles, which the child referred to as sixths, were the same size as one diamond, which the child referred to as a third. Although these relationships are true if the hexagon is considered the whole, the students who have a pieces perspective do not address the relationship between the piece and the whole.

It was most evident that students had a pieces perspective when they drew pictures of fractions based on a pieces model. For example, on the mid-interview Bobby said he did not know if 2/3 and 4/6 were the same, but he thought they might be because “three plus three equals six.” He decided to draw a picture to explain his answer, which is in Figure 1. He drew a large rectangle and divided it into half and called it two-thirds. Based on how he drew the picture, Bobby appeared to be more concerned with making the two pieces, and he just called the pieces thirds. He drew a second rectangle that was the same size and adjacent to the first rectangle. He divided into half, divided each half so it had four pieces, and said it showed 4/6. Again he focused on the number of pieces in the numerator and did not make an attempt to connect sixths to the whole. Not only did he assume that the fractions were equivalent based on the way he drew the picture, his drawings focused on making the pieces specified in the numerator without regard to the whole.

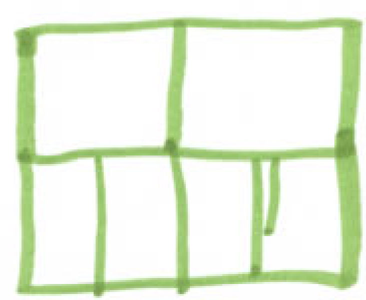


Figure 1: Bobby’s Drawing of 2/3 and 4/6

On the post-interview, I asked students to compare 2/3 and 3/4 in one problem and 2/8 and 3/12 in another. Several students chose to use the fraction kit. Once they laid out the pieces from the fraction kit, they saw that 3/4 was larger than 2/3 and that 2/8 and 3/12 were equal. The students used the materials to find and support their answer, but these particular materials enabled them to do so without reference to the whole. Although it was not as important for this particular problem because the wholes were the same, it could create problems in situations where the wholes were different. As part of my interview protocol, I asked these students if they could think of another way to prove that 2/8 and 3/12 were the same amount without using the fraction kit. As on the mid-interview, Bobby demonstrated his strong reliance on the pieces perspective by drawing two small rectangles and labeling each one as 1/8 (see Figure 2). He then tried to draw three smaller rectangles above to show the 3/12. These rectangles resembled the fraction strips he used to solve the problem initially. As demonstrated through Bobby's work, his pictures were of fraction pieces and did not include the whole. It was apparent that he was either not aware of or did not pay attention to the relationship of the fraction to the whole.

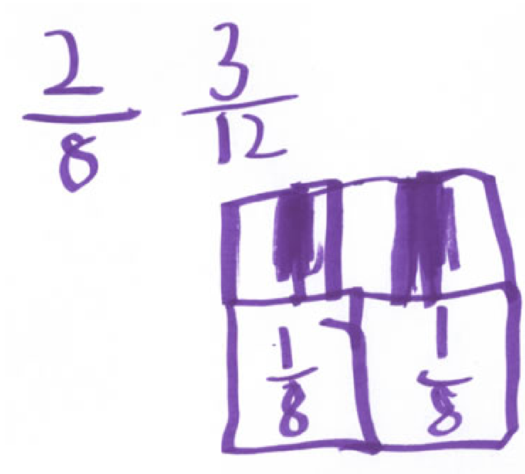


Figure 2: Bobby's Drawing of 2/8 and 3/12

Ten of the students used a pieces perspective to compare and order fractions on at least one problem, primarily on the mid- and post-interviews. This perspective was more apparent when students were drawing pictures than when they were using the fraction strips or pattern blocks. The use of the physical models was not sufficient to distinguish between the pieces and part-whole perspectives in terms of children's underlying understanding of fractions. Since the fraction kit allowed the pieces to be manipulated independently from the whole, it is not clear whether students did or did not understand the relationship between the fraction pieces and the whole. Therefore, some students who were using the pieces from the fraction kit may have had a part-whole perspective.

PART-WHOLE PERSPECTIVE

Whereas students with a pieces perspective viewed fractions as absolute amounts, students with a part-whole perspective focused on the relationship of the parts to the whole. The key relationships that students attended to continued to evolve as their understanding of fractions developed. Students started with the whole and used the denominator to determine how many parts were in the whole. As students developed more complex relationships, they observed or created additional parts through repeated halving. The most advanced students used recalled facts to make equal-sized pieces that maintained fractional relationships. For example, students used the multiplication fact that 2 x 3 = 6 to make sixths by first making halves and then dividing each half into three additional pieces.

Students with a part-whole perspective frequently used drawings to order and compare fractions. Often they started with a whole and partitioned it into the number of pieces needed for a specific denominator and then identified the amount in the numerator. Many students divided circles into halves with one line and into fourths by adding a second line, forming a "+" sign in the middle of the circle. To make eighths, students added two more lines forming an "x" in the circle. Although students were making partitions using relationships to draw these common fractions, they did not seem to be aware of the relationships. It was apparent that the number of pieces was the primary concern when they then added one line to the fourths to create six pieces and two lines to the eighths to make twelve pieces. The students continued to add lines and count the number of parts in the whole and did not worry about the different-sized pieces. Christina tried to use this approach to draw pictures to compare 2/3, 6/9, and 8/12 during the pre-interview (see Figure 3), but she was not sure how to divide circles into three and nine equal-sized pieces. She switched from circles to long skinny rectangles and divided each into the correct number of pieces by drawing lines to partition the rectangle moving from one end to the other end. She appeared to visually compare the amount of the whole that was shaded in for these problems. She also drew a rectangle to compare 8/12 and decided it was also equal. As this example demonstrates, students can understand that fractions are part of a whole without addressing equal-sized pieces or relationships within their representations.

Figure 3: Christina Drawing of 2/3, 6/9 and 8/12

At another level of complexity in the part-whole perspective, students were concerned about having equal-sized pieces or equal-sized wholes, but some students only attended to one of these two features at a time. Christina attempted to draw pictures to compare the fractions 10/12 and 5/6. She explained, “I thought it was really hard to draw sixths on a pie graph so I tried a bar. But then I realized that it was really hard for me to try to get it exactly right – to put the lines where they're supposed to be.” Even though Christina wanted to make equal-sized parts, she did not have a strategy to do so. To create equal-sized wholes, another student tried to trace her original circle on the back of her paper. When she was unable to compare the two fractions back to back on her paper, she decided to trace a hexagon pattern block to make equal-sized wholes. Although she was concerned about having equal-sized wholes, she did not try to make equal-sized parts. During the interviews equal-sized wholes tended to be less of a concern than equal-sized pieces. When students did not know how to create equal-sized pieces by using relationships such as dividing halves into three pieces to make sixths, they used a trial and error approach or solved the problem using a different perspective.

At the third level of complexity, students started thinking about how to make equal-sized pieces to compare and order fractions. The recognition of these relationships was emergent in nature. When asked to compare 5/6 and 10/12, Mark struggled with how to make twelfths. He drew a circle, divided it into eighths, and then cut four of the eighths in half (see Figure 4). There were twelve pieces that were not equal to each other. Since he had talked about wanting to make the pieces equal when he was dividing the circles, I asked him about his picture for twelfths. Mark was uncertain if the pieces were equal so he drew his picture again by first making eighths and said that the pieces were equal now. Then he realized that he had a problem, "Because if I halve one, that will be uneven to the rest." He divided all of the pieces and halves and counted the pieces to see that he had made sixteenths. So although Mark was not able to draw twelfths, he figured out that his method of drawing twelfths did not result in equal-sized pieces. He also recognized that his repeated halving strategy created equal-sized eighths and sixteenths. Although Mark observed these relationships while making his drawings, other students planned ahead using number relationships.

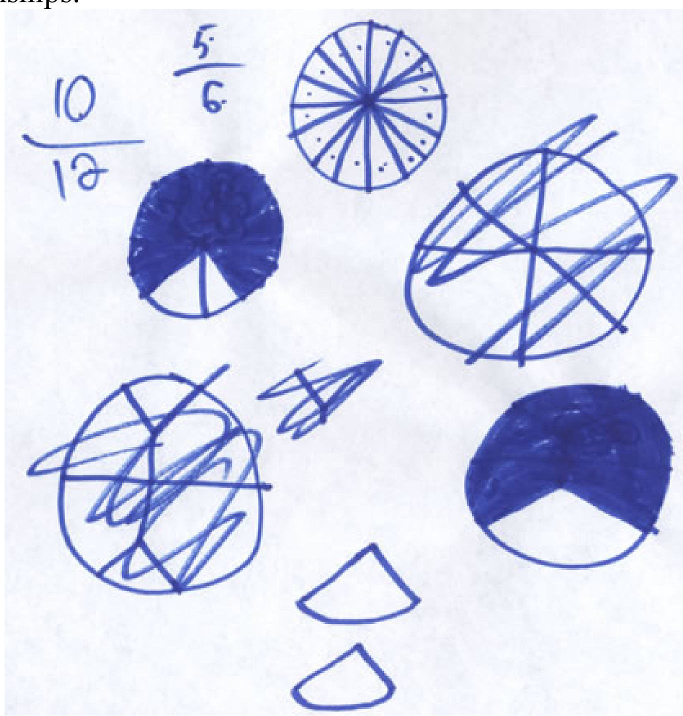


Figure 4: Mark's Drawing of 5/6 and 10/12

When students used number relationships to plan their drawing, their representations with equal-sized pieces allowed them to compare and justify their answers fully. Elisabeth and Marie used multiplicative relationships to compare 2/3 and 6/9. Elisabeth used the fact 3 x 3 = 9 to draw a picture for 6/9 by first making thirds and then dividing each third into three pieces (see Figure 5). She used the same approach to determine that 12/18 was also equal to 2/3 and 6/9.

When I asked Marie how she knew to make the ninths, she explained that, "Three times three equals nine. Three little pieces would be one three, three more pieces would be two threes, and three more pieces would be three threes." Unlike many students who struggled with making ninths, both of these students used known relationships to plan how to draw their representations.

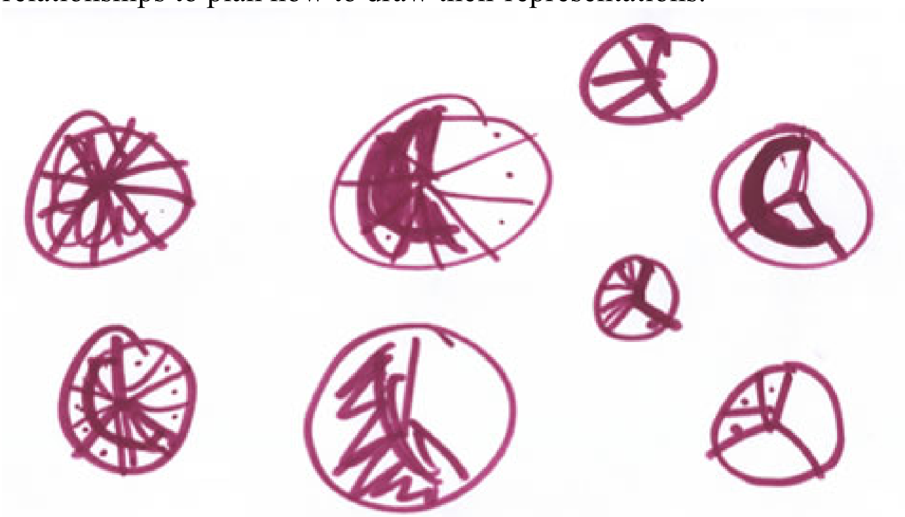


Figure 5: Elisabeth's Drawing of 2/3, 6/9

Twelve of the 13 students in this study solved at least three problems using a part-whole perspective. The thirteenth student, Bobby, only used the part-whole perspective for one problem to show the relationship between 1/2 and 2/4 and was unable to represent 3/6. One plausible explanation is that Bobby had very strong pieces and unit fractions perspectives and did not understand the importance of the whole for comparing and ordering fractions. Although this perspective was most common on the pre-interview, students frequently used it on the mid- and post-interviews. Even though students frequently relied on physical materials to solve problems using a part-whole perspective, the unit fraction perspective allowed students to start using fractional relationships without the physical representation.

UNIT FRACTION PERSPECTIVE

Students who had a unit fraction perspective focused first on the unit fraction. Many of these students recognized the key relationship that the number of pieces in the whole determined the size of the unit fraction. Although understanding the inverse relationship between the size of the denominator and the size of the pieces was most likely derived from a part-whole perspective, students did not always make this connection explicit. When explaining answers, some students referred to physical materials while others did not. Another important relationship for some students was that a composite fraction was a unit fraction iterated a certain number of times.

The simplest level of the unit fraction perspective was confined to fractions with the same numerator. When they were comparing unit fractions, these students realized that the smaller the denominator, the larger the unit fraction. For example, one student explained 1/5 was the smallest "because it is divided into more parts" and 1/3 was the largest "because it is divided into less parts." On the other hand, some students only focused on the numerical relationship, which another student explained as, "The bigger number is smaller and the smaller number is bigger." He recognized the inverse relationship between the size of the numerator and the size of the piece, but did not make the connection between the observed relationship and the physical materials explicit in his explanation. When ordering fractions with the same numerator, students used a similar strategy.

This process became more complicated when they were comparing non-unit fractions with different numerators. During a class discussion, Mark explained that 4/7 was larger than 3/8 because 1/7 was larger than 1/8 and there were more sevenths. Several students with a unit fraction perspective tried to use this strategy for multiple problems. On the post-interview, Bobby compared 7/15 and 6/10 using “the higher the number goes, the smaller the fraction gets.” He went on to explain, “This is 6/10 and that's almost the same as 7. And so, 10 is way smaller than 15. Not way, but smaller. So I know that fraction [pointing at 6/10] is bigger than that fraction [pointing at 7/15].” Bobby was making a qualitative judgment about the relative differences between the size of the unit fractions and the size of the numerators. Since the numerators were only one apart, Bobby reasoned that the larger size of the tenths made up for having one less piece. Although this worked for some problems, it was an insufficient perspective for comparing other fractions such as 3/5 and 4/6. Bobby decided that these fractions were equal because, “Six is one more than 5 so that would be a smaller fraction, but it has 4 and that only has a 3.” This strategy built on unit fractions allowed Bobby to use qualitative reasoning to make estimations about the relative size of some fractions, but it was not adequate for both problems. Mark used similar logic to determine that 6/10 was larger than 7/15 and that he did not have enough information to compare 3/5 and 4/6. Being able to determine when this strategy provided sufficient information to order fractions demonstrated the fullest understanding of comparing fractions using a unit fraction perspective.

All 13 students used the unit fraction perspective to solve at least one problem, primarily when the fractions had the same numerator. Four students tried to use the unit fraction perspective to help them solve problems with different denominators on more than one occasion with varying success. Bobby, the student who only used the part-whole perspective once, focused on the unit fraction repeatedly with limited success. Mark, who tended to use a part-whole perspective frequently, was more successful in that he could identify when he did not have enough information to compare the fractions. Although some students referred to the physical models when describing the size of the piece, students tended to think about the fractions in relationship to the unit fractions. Students with a within-fraction perspective further connected numerical relationships with comparing and ordering fractions.

WITHIN-FRACTION PERSPECTIVE

When students focused on the relationship between the numerator and denominator, they had a within-fraction perspective. The primary relationships that students described between the numerator and denominator were fractions equal to 1/2 and 1. One strategy from the within-fraction perspective was to use the multiplicative relationship between the numerator and denominator to prove that two fractions were equivalent. This relationship was used by students in reference to 1/2.

Another strategy was to introduce a third fraction such as 1/2 or 1 to make a comparison. This required an implicit use of the relationship between the numerator and denominator. Students realized that when the numerator and denominator were the same, the fraction was equal to 1. Several students explained that since 4/3 was greater than a whole, it was larger than 3/4 which was less than a whole. Building on this understanding, students extended these relationships to convert between improper fractions and mixed numerals without explicit instruction. When they wanted to figure out what was half of a fraction with a specific denominator, students usually began with the denominator and divided it by two. Students were comfortable with making the numerator a fraction to identify fractions equivalent to 1/2 such as “three and a half sevenths” or “seven and a half fifteenths.”

The within-fraction perspective was only occasionally used to prove fractions were equivalent. For example, when asked to compare and generate fractions equivalent to 1/2, several students explained that the top number was half of the bottom number as they suggested fractions such as 4/8, 5/10, 8/16, and 50/100. Marie used the multiplicative relationship when comparing 2/8 and 3/12 as she explained that, “2 is 1/4 of 8 and 3 is 1/4 of 12, so they're the same. They are both equal to 1/4.” It is possible that using within-fraction relationships for equivalent fractions is limited to very specific groups of fractions such as those equal to 1/2, 1/3, or 1/4.

Students primarily used the implicit relationship between the numerator and denominator to compare fractions. For example, Cheyenne demonstrated her tenuous understanding of fractional relationships when she was comparing 4/6 and 3/5 to 6/6 and 5/5 respectively. At first she said, “This (4/6) is the biggest. Because you just need two more sixths and you need two more fives,” and then she said 3/5 was larger. After identifying how much more was needed to make a whole, Cheyenne attempted to compare the missing amounts but was unable to use all of this information and determine that 4/6 was larger because it had the smaller amount missing. Christina used a more intuitive approach to compare these fractions to 1/2. After deciding that 4/6 was larger, she justified her answer by explaining, “Because it's [pointing at 4/6] over half of six and this one [pointing at 3/5] is right about half.” Christina did not make the within-fraction relationships as explicit as another third grader who explained, “Seven-fifteenths is lower than half and 6/10 is bigger than a half.” He elaborated that 7/15 was less than a half, “Because seven and a half is half of 15, which would be the half point.”

When students compared fractions that were not close to 1/2, they continued to use it as a starting place. Students used a variety of strategies to explain how they figured out that 2/7 was smaller than 7/15 and 6/10. Allison stated that 2/7 was closer to zero than the other fractions, and several students used reasoning similar to Todd who explained, “Two-sevenths is the least because it's not even at the half way point and those are.” Two other students initially identified three and a half sevenths as a starting place to quantify 2/7 as close to 1/4 and 1/3. Since the fraction 2/7 was not particularly close to any of the typical benchmark fractions, students compared it to 0, 1/2, 1/3 and 1/4 and justified their answer.

The within-fraction perspective was used by all 13 students. Students used this understanding to help them solve multiple problems over the course of the unit, especially on the post-interview. This might have been influenced by the students' experiences with working with benchmarks during the latter half of the unit or the specific numbers chosen for the post-interview. The use of benchmark fractions was a powerful and efficient strategy for these students who examined the relationship between the numerator and denominator flexibly. By using a within-fraction perspective, students solved a variety of comparison fraction problems without using traditional procedures for comparing and ordering fractions. Whereas students tended to use the within-fraction perspective to solve comparison problems, they primarily used the between-fraction perspective to solve problems with equivalent fractions.

BETWEEN-FRACTION PERSPECTIVE

In contrast to examining the relationship between the numerator and denominator in the within-fraction perspective, students with a between-fraction perspective focused on the relationships between numerators and denominators of different fractions. Both multiplicative and additive reasoning were important for students with a between-fraction perspective. Students first recognized that doubling or halving the numerator and denominator generated equivalent fractions. Although students stated that they were "doubling" the numerator and denominator, it is unclear whether they multiplied the numerator and denominator by 2 or added the numerator and denominator twice. Some student specifically explained that they were multiplying the numerator by 2 and multiplying the denominator by 2. Students in this study with a between-fraction perspective did not extend this relationship to tripling. When students identified a fraction that was a result of tripling, they focused on additive relationships before multiplicative ones. Although many additive strategies are not mathematically valid, some students in this study developed additive strategies that maintained the ratio relationship between the numerator and the denominator.

The between-fraction perspective was used on a few different problems, but was used most frequently to compare and generate equivalent fractions for the following problem:

> The children in Ms. Lee's class were solving an equal sharing problem, where all the pancakes were the same size. Some got 2/5 of a pancake for their answer. Others got 4/10 of a pancake for their answer. Are these the same amounts of pancake or different amounts of pancake? What's another way to say that amount of pancake with fractions? Could 6/15 be the same amount as 2/5 and 4/10?

This particular problem with examples of student work illustrates the different types of relationships students addressed when they focused across numerators and across denominators.

Mark decided that 2/5 and 4/10 were the same and explained, "Well 5 is half of 10 and 2 is half of 4." When asked to prove his answer, Mark tried to draw a picture for each of the fractions, but he was unable to use the relationships he described to assist in his drawing. He doubled the numerator and denominator to generate other equivalent fractions such as 8/20 and 16/40. When I asked if 6/15 could also be equivalent, he answered, “I don't know if that would be the same or not. I don't know how to make fifteenths, so I don't know how to find out if that's the right answer or not.” Mark’s understanding of the between-fraction relationships only applied to doubling and did not extend to tripling, and the part-whole perspective continued to play an important role in proving answers.

A couple of students multiplied the numerator and denominator by 2 separately. One student decided that 8/20 was equal to 4/10 and explained, “Basically I just used the 4/10 and I multiplied the 10 by 2…. And so I multiplied the 10 by 2 and I got 20. Multiplied the 4 by 2 and I got 8.” Another student used similar reasoning to claim that 2/5 was equal to 4/10 because “Because 4/10, let's see how, 2/5 - if you multiply the nominator [sic], which would be 4, and then you multiply the denominator, it's going to be 10, so it's 4/10.” Although these students used a between-fraction perspective by working with whole number relationships and not referring to the procedure of multiplying by 2/2, they were almost at the point of using a transform perspective.

Building on the problem comparing 2/5 and 4/10, three third graders found relationships across the numerators and denominators that helped them determine 6/15 was also equal to the other fractions. Joey used the 8/20 that he had already figured out. He used the facts $4 + 4 = 8$ and $10 + 10 = 20$, and then added only half as much for the second number so that he had $4 + 2 = 6$ and $10 + 5 = 15$. Although he saw this relationship, he said, “I'm still not really sure if that's the right answer.” Danielle had written 2/5, 4/10, and 6/15 across her paper and noticed a pattern. She wrote, “You skip count by 5's at the bottom and 2's at the top." Although this was a valid pattern, Danielle could not explain why it worked. After using doubling to decide that 2/5, 4/10, and 8/20 were equivalent, Christina was not sure about 6/15 because it could not be obtained by doubling like her previous answers. Then she said, "It would be right because it's just adding 2 to this [pointing at the 4] and it's adding 5 to this [pointing at the 10]." Even though Christina correctly identified the ratio relationship in her explanation, she focused on adding whole numbers and considering relationships between whole numbers. She added, "Because this is divisible by 5 [pointing at the denominator for 4/10] and this is divisible by 2 [pointing at the numerator for 4/10]." Christina's reasoning, based both on additive and multiplicative relationships, allowed her to explain her answer using more complete mathematical reasoning than her classmates. All of these students maintained the ratio relationship by adding ratio units across numerators and across denominators.

When students used a between-fraction perspective for solving problems, they focused on the relationship across denominators and numerators. This was a distinct approach for solving equivalency problems, but it was not a very common approach. Only 8 of the 13 students used this perspective to solve problems. It is possible that the next more frequently used perspective permitted students to solve other equivalence problems more efficiently.

EQUIVALENCE PERSPECTIVE

When students had an equivalence perspective, they focused on equivalent fractions. Relationships between equivalent fractions were extremely important as students solved order and equivalence problems. Students often started with "recalled facts" that were tied to their experiences working with manipulatives. Some students identified the relationship between a unit fraction and the fractions that resulted from dividing the unit fraction in half. Using recalled and derived equivalent facts, students extended the relationships to different fractions.

During the first half of the unit, students worked with a fraction kit that included, halves, fourths, eighths, and sixteenths. On the mid-interview students frequently used equivalent relationships in the fraction strip kit to solve problems. When I asked Bobby how many sixteenths were in 2/8, he quickly answered "4" without using physical materials. He explained that he knew, "2/16 equals 1/8. 1/8 plus 1/8 equals 2/8 and 2/16 plus 2/16 equals 4/16." When I prompted Christina to think of another fraction equivalent to 1/4, she said 4/16 without touching the physical materials. Not only were these relationships used to find equivalent fractions, many students used the relationship 1/8 equals 2/16 to determine that 5/8 was larger than 3/16. When I asked Christina how she figured out the answer without using the physical materials, she explained, "Because I knew that 2/16 was 1/8 and I knew that – so I only had 3/16. I knew it couldn't be 5/8." Some students described 3/16 as only 1/8 and a half of an eighth, while others described it as 1/8 and 1/16. Many of the students explained that they remembered these equivalent relationships from playing games with the fraction strips.

Additional fraction pieces including thirds, sixths, and twelfths were constructed during the second half of the unit. Again students identified equivalent fractions based on these physical materials when they were comparing fractions. On the post-interview, Joey was ordering the fractions 2/3, 3/4, 1/6, 1/2.

He correctly placed them in order without a picture, symbols or concrete materials. When asked how he solved the problem, he explained the fractions in terms of sixths, “I know that 1/2 is 3/6 and 3/6 is bigger than 1/6. And I know that 2/3 is bigger than 1/6 because 2/3 is 4/6. And 4/6 is bigger than 1/6.” He also said the 3/4 was bigger than 2/3, and he justified his answer by saying, “Because 3/4 is [pause] is 9/12 and 2/3 is six, wait, 8/12.” When asked how he knew 3/4 equaled 9/12, Joey continued, “Well I know my, that… well I tried it. I tried 3/12 on 1/4. I tried as many twelfths as I could fit on 1/4 and I got 3/12.” Only at this point did he take out the fraction kit to demonstrate the equivalent relationship between 3/12 and 1/4. As several students did on the interviews, he focused on equivalent relationships that were strongly connected to the physical material.

Students extended these equivalent relationships to fractions that did not have a specific piece within their fraction kits. Some students focused on the fraction that resulted by dividing a unit fraction in half. As Mark explained, “Six is half of 12, so 2/12 put together would be just as big as a sixth. If you take a sixth and halved that, it would be a twelfth.” When asked to compare 2/3 and 4/6, Mark extended the equivalent relationship by explaining, “They are the same amounts because 1/3 is 2/6, so that would mean 4/6 is 2/3.” Similarly, Joey stated, “I know that if you divide a fifth into tenths, you get 2/10 because you're dividing. If you are dividing one fraction, you divide it into 2 other pieces, into 2 pieces. And so that would be 2/10 would equal 1/5 and so 4/10 would equal 2/5.”

Christina extended relationships to include fractions equivalent to 1/2 and 1/4 on the mid-interview. At this point the class had been using the fraction kit with the halves, fourths, eighths, and sixteenth pieces, but they had not made the other fraction pieces. I included a question comparing 2/8 and 3/12 knowing that they had a physical model for only one of these fractions. Christina was talking quietly to herself as she used the fraction kit and placed the 1/4 piece on top of the 1/2 piece. She then laid two 1/8 pieces on top and then said they were the same. She knew that they were the same because she knew that 1/4 was 3/12. She explained, "I know 1/2 is 6/12. And I know that a half of a half is 3/12. So I knew that a fourth is a half of half, so I put a fourth on here," as she pointed at the 1/4 piece on the half. Since Christina did not have the fraction pieces available in her kit, she figured out equivalent relationships beyond the physical materials.

All of the students used the equivalence perspective at least once. Although students used it some on the pre-interview, it was more common on the subsequent interviews. One of the observations was that when students referred to a physical material to explain an equivalent relationship, they only mentioned fraction strips. One possible reason that students focused on equivalent relationships more on the mid- and post-interviews is that the use of the physical materials supported the development of this perspective. Students definitely understood that there were multiple equivalent fractions and used these relationships effectively to solve problems. Students observed, generated, and relied on the equivalent relationships as indicated in their justifications. The relationships in the transform perspective were not as apparent.

TRANSFORM PERSPECTIVE

When students had a transform perspective, they focused on using rules to compare and order fractions. Although this was not the primary focus of the fraction unit in this class, a few students compared or generated equivalent fractions symbolically following a rule of multiplying or dividing by a fraction expressed in the form n/n. A few students converted to a common denominator to order specific fractions. Some students implemented a rule but could not explain why the procedure resulted in a correct answer. Three fifth graders were the only students to frequently use the transform perspective to compare and order fractions. Each of them articulated why the procedure worked during at least one of the interview questions, but they each did so in a different way.

On the pre-interview, Elaine explained that 2/3 and 6/9 were equal and then justified her answer by explaining the commonly taught procedure, which she said she learned from her mother. Elaine explained, “3 x 3 is nine and 2 x 3 is six. And if the numerator and denominator are the same, it's - it can be the same number.” When asked why this worked, she replied, “If I multiply by 3/3 - that is actually 1 – I have to multiply the numerator by something to get to another number and the denominator by something to get to another number, but the number in the middle has to be equal to 1.” She used a similar approach to analyze 8/12 by first writing 2/3 x _/_ = 8/12 and realized that 2 x 4 = 8 and 3 x 4 = 12 so it was true. Elaine used the same procedure and justification on the post-interview when comparing 2/5 and 4/10. When asked how she could explain this to a classmate who did not understand this procedure, she was very quiet as she thought about it. Then she rationalized, “I'm multiplying it - 2/2 is equal to 1. Well, like if you use whole numbers 4 x 1 = 4, so it equals the same thing. But when you use fractions, if you do times, you multiply the denominator times denominator. 5 x 2 = 10 and 2 x 2 = 4. And I don't know why that works.” Elaine’s understanding of the procedure for generating equivalent fractions was clearly tied to multiplying by 1.

Elisabeth used the transform perspective to prove that fractions were equivalent and to order non-equivalent fractions. When asked to order the fractions 2/3, 3/4, 1/6, 1/2 from least to greatest, Elisabeth realized that “all above go into 12.” After converting all of the fractions into twelfths, the only student in the study to do so, she ordered the fractions in the form of twelfths and then wrote the original fractions. She verbalized the steps to justify her answer. Although this was not an efficient method for solving this problem, it was a valid strategy and demonstrated her comfort level with using this procedure. On another problem comparing 2/5 and 4/10, Elisabeth said the fractions were the same and described the procedure of multiplying both the numerator and denominator by 2/2. When I asked her why doing the same things to both the top and bottom numbers, she used a concrete example to illustrate her reasoning: “Because if some people are sharing, like if five people are sharing two things, you multiply the people times 2. They'll get half as much. Yeah, half as much. But then of course, if you multiply the things that they are sharing times 2, they'll get the same amount as they were.” Elisabeth’s explanation relied on the part-whole perspective that was developed earlier in the unit through solving equal-sharing problems.

Marie used a transform perspective for justifying equivalent relationships and generating new equivalent fractions throughout the interviews. Although she correctly used this procedure for solving problems, she was unable to provide a mathematical justification. On the pre-interview Marie quickly recognized that 2/12 and 1/6 were the same amount and demonstrated this by dividing both the numerator and denominator from 2/12 by 2. When I asked why dividing by 2 worked, she explained that you could use other numbers like 3, 4, or 5 but did not explain why. At the end of the unit, Marie clearly explained how the procedure produced an equivalent fraction through her drawings for 2/5 and 4/10. In addition to using the rule for generating equivalent fractions correctly, she understood why the equivalent relationship was maintained. Although Marie solved problems using a transform perspective, she used the part-whole perspective to justify her answers in a more concrete manner.

Although procedures are commonly taught for comparing and ordering fractions, the transform perspective was only used to solve problems by 5 of the 13 students. This was related to the fact that the teacher did not emphasize using procedures. There were only a handful of students who relied on using procedures to solve problems, and these tended to be the students who were consistently performing well above average in this class. These few students introduced procedures to the rest of the class for finding a common denominator and multiplying or dividing by n/n during whole class discussions when students shared how they solved problems. Some of the students tried the strategy they had seen modeled in class of multiplying the numerator and denominator by the same number, but they could not explain why it worked. Other students provided a stronger justification, especially when the reasons were based on other perspectives and a deeper understanding of fraction concepts.

SUMMARY OF PERSPECTIVES

These perspectives emerged through student interview questions related to order and equivalence. Although each perspective is distinct because students focused on different aspects of fractions, they are also interrelated. When students recognized the inverse relationship between the number in the denominator and the size of the piece, they probably developed this understanding by solving problems using a part-whole perspective. The equivalency perspective seemed to be directly related to students' observations of equivalent relationships from using concrete materials. When I asked students to justify answers when they used procedures, they moved away from the transform perspective to do so.

These perspectives provide only a portion of the findings from this study. The interconnections between these perspectives, the use of physical models and children's developing number sense related to comparing and ordering fractions are all integral to the findings of this study. The following chapter examines the themes that arise from these interconnections.

Chapter 5: Themes

The themes in this chapter build on the perspectives described in chapter four. Perspectives were based on children's approaches to solving problems, especially the relationships that they focused on as they solved problems. Many of the themes connect the perspectives with the role of the physical models in developing children's understanding of order and equivalency with fractions. Children's perspectives and use of physical models directly influenced children's understanding of relative size and the development of number relationships. Students who were developing number sense about fractions made judgments about effective ways to solve problems and alternative ways to justify their answers as they moved fluently between multiple perspectives.

THEME ONE: IMPACT OF PRE-PARTITIONED PHYSICAL MODELS

Area and/or linear pre-partitioned physical models can positively impact children's development of number relationships for comparing and ordering fractions when students examine the relationships between wholes and different parts and are able to extend these relationships to fractions not included in the physical models.

Although area and linear models are continuous models, children tended to treat physical models that were divided into a certain number of pieces as though they were discrete. Since the physical models were already subdivided, students were not as likely to continue to make partitions. During this study, the two physical materials that fit this description were the pattern blocks and fraction strip kits. Pattern blocks were used as an area model with consistent relationships between the triangles, diamonds, trapezoids, and hexagons. Even though some of the activities varied what was considered the whole, students usually considered the hexagon as the whole when they were solving other problems. Students constructed the other physical model, fraction strip kits, during class. Students started with the whole for making all of the pieces and folded and cut all of the pieces. Ms. Bell encouraged them to use relationships to fold and cut so they would make equal-sized pieces. Due to the dimensions of each strip, the kit could be considered either an area or linear model. Most students preferred to solve problems using their fraction kits over the pattern blocks even though both models were readily available in the classroom and during interviews.

This major theme is comprised of three subthemes that describe specific ways that these physical models, which students treated as discrete models, hindered and helped students compare and order fractions. Some students began to view fractions as absolute amounts and this limited their ability to connect specific fractions to the whole. When students used physical models to solve a problem, they focused on just figuring out the answer and did not examine number relationships. When the students could not use the physical model to figure out the answer, the physical models supported students in identifying and remembering equivalent relationships. Each of these subthemes is examined in the following sections.

Subtheme A: Fractions as Absolute Amounts

Pre-partitioned physical models allow students to think about fractions as absolute amounts and the relationship between the size of the fraction relative to the whole is obscured.

When comparing and ordering fractions using the pre-partitioned physical models, students often represented the fractions without using the whole. For example, to compare 1/6, 1/2, 2/3 and 3/4, students represented each of the fractions using the correct number of each-sized piece and compared the length or area of each fraction. Even though students were not using the representation for a whole in the comparison, some students indicated that the fractions were related to the whole during their explanations or alternative strategies. On the other hand, some students did not seem to be aware of the size of the whole.

Bobby, a fourth grader, had developed an understanding of fractions as pieces. Two examples of him comparing 2/8 and 3/12 demonstrate his focus on the fraction pieces. During the mid-interview I asked him to compare these fractions when he did not have twelfths in his fraction strips kit. He attempted to use the eighths from his current fraction kit and then twelfths from an area model made from a large index card fraction kit the previous year. When I asked how the two kits were different, Bobby replied, “The kit from this year doesn't have twelfths and the kit from last year doesn't have eighths.” This response indicated that he continued to focus on the pieces. Next, I asked if the wholes from each of the kits were the same size. He placed the whole from the current linear fraction kit on top of the large index card and measured one section before deciding they were not the same. When I asked if that affected how he should compare the fractions, he answered, “Maybe. Probably.” Even though my questions guided him to think about the relationships between the wholes, Bobby did not have a clear understanding of the importance of the whole.

When asked to compare these fractions on the post-interview, Bobby used the fraction strips kit that now contained twelfths and decided that the fractions were equivalent. For his second strategy, Bobby drew one rectangle, then a second one adjacent to it, and wrote 1/8 in each rectangle. He treated the representation of 1/8 as an absolute amount based on the pieces from the fraction strip kit. Since Bobby knew that a twelfth was not as big, he drew a smaller rectangle above the first 1/8 piece. He drew the second 1/12 piece on the far right side, and then connected the two sections by drawing a straight line. After ending up with a wide piece in the middle, Bobby tried making some adjustments to the other two pieces. He used the drawings to illustrate the relationship between 2/8 and 3/12 that he found with the fraction strip kit. He continued to consider fractions as pieces and did not consider how the fraction pieces were related by the whole.

There is additional evidence that this understanding of fractions was supported by the use of the pre-partitioned physical model. Although Bobby frequently used a pieces perspective to solve problems on the mid- and post-interviews, he never used it on the pre-interview. Only one student used a pieces approach on the pre-interview, while it was more common on the subsequent interviews after these pre-partitioned physical models had been used in class. Another observation is that although the part-whole model was by far the most common perspective for comparing and ordering fractions, Bobby only used it once during all of the interviews. He used it to show the relationship between 1/2 and 2/4 on the pre-interview, but could not extend his drawing to show another fraction such as 3/6 or 4/8. Based on Bobby's approaches to solving problems, it seems that he was developing an understanding of fractions as absolute amounts, and this obscured his understanding of the importance of the whole in determining the size of the fractions.

Subtheme B: Physical Models Discourage Number Relationships

The use of the pre-partitioned physical models can discourage children from developing and using number relationships to solve problems.

One reason that the pieces perspective was probably used more frequently on the post-interview was due to the fact that two of the problems could be solved using the fraction strip kit. Students reached for the fraction kit, and used the pieces to get answers for some of the questions. When students did not have representations for the fractions in the problems during the interview, they tended to use other relationships for comparing and ordering fractions. During class, students sometimes attempted to make additional the fraction pieces. The presence of the physical model encouraged students to use it as a tool to get an answer instead of focusing on number relationships.

This difference can be demonstrated by how Christina compared 2/8 and 3/12 on the mid- and post-interviews. On the mid-interview, the students did not have a representation for twelfths in their fraction kit. Using number relationships, Christina reasoned that since 6/12 was equal to 1/2, 3/12 was equal to 1/4. She demonstrated that 2/8 was also 1/4, so she knew that 2/8 and 3/12 were equivalent. On the post-interview when she had representations for both fractions in her kit, she chose to show the representation with her kit. When I asked her for a second way to explain why they were equal, she said, "They're the same because 1/3 is equal - no wait - equal to 1/4." She demonstrated this by placing the 1/4 piece over the 2/8 and then added the 3/12 on top. Her use of number relationships was much more sophisticated on the mid-interview when she did not have a representation for twelfths.

Another example was when students compared 4/7 and 3/8 during a class warm-up activity. Ms. Bell presented this problem after students had done several other activities on previous days using benchmark fractions. She wanted students to recognize that 4/7 was greater than a half and 3/8 was less than a half, so 4/7 was larger. Most students did not begin by looking for relationships. Some students decided to make sevenths for their fraction strip kit first. As Allison determined using a calculator and dividing the length of the whole by 7, each piece had to be "A little more than 2 1/2. It's close to 2 60/100." Other students were drawing pictures and using a ruler to figure out the size of the pieces. A couple of the more advanced students converted to a common denominator. When Ms. Bell brought the class together, she asked the students:

> Think of something different. I want you to look at 3/8 and 4/7, and I want you to think about what you already know about fractions so that you can compare those without drawing a picture, without using the fraction kit, without using common denominators. Is there anything you know about fractions that can help you?

Based on this prompt, students began to look at the relationship between sevenths and eighths, a unit fraction perspective. Finally, after Ms. Bell reintroduced the term “benchmarks” and asked the students to describe it, they solved the problem using this strategy.

Even though a CGI approach for teaching mathematics focuses on number relationships, students became so focused on comparing the fractions using physical representations, they did not look for number relationships. In this case it required teacher guidance to move the students away from the physical models for fractions to think about other relationships.

Subtheme C: Development of Equivalent Relationships

The pre-partitioned physical models can encourage students to examine equivalent relationships between different fractions and extend these relationships to fractions not included in the physical model.

The development of the equivalence perspective was related to students work with the fraction strips. Many students remembered equivalent relationships between halves, fourths, eighths, and sixteenths and drew upon these relationships as recalled facts when they were solving problems. Sometimes students started to use the fraction kit, but after only putting a couple of pieces down, they solved the problem based on relationships. When asked about how they knew certain relationships, students often referred to using the fraction kit and playing the cover-up game.

Although Bobby repeatedly used a pieces perspective to solve problems, he also used the physical model to help him with equivalent relationships. I asked Bobby to combine the fractions 1/2 and 1/12 at the end of the unit. Bobby took out the fraction kit. He placed a whole on the table and then put 1/2 and a 1/12 piece above it. Without adding more pieces, he answered 7/12. When I ask how he knew 7/12, he explained, “Because 6 - there's 12/12 make a whole, so half of 12 would be 6/12. And so 6/12 and I had one more so I added it on to make 7/12.” The only other two times Bobby used an equivalence relationship he remembered that 1/8 was equal to 2/16. For a student who struggled with number relationships in fractions, the few relationships he used were directly connected to the physical models.

The process of actually making the strips, by taking a fraction piece and folding it and cutting it in half, might have helped them build the relationship between a unit fraction and the resulting fraction when it was divided into two pieces. There were multiple times that students used this relationship to explain their answers. Allison explained that “2/12 equals 1/6 as 2/10 equals 1/5.” When asked to explain how she knew the relationship between 2/12 and 1/6, she folded a piece of paper into six pieces and then folded it in half again. She explained that the 1/6 was the same amount as the 2/12. So even though students had not made fifths, tenths, sixths, or twelfths at the point this question was asked, Allison extended her reasoning beyond the manipulatives.

THEME TWO: RELATIONSHIPS IN THE PART-WHOLE PERSPECTIVE

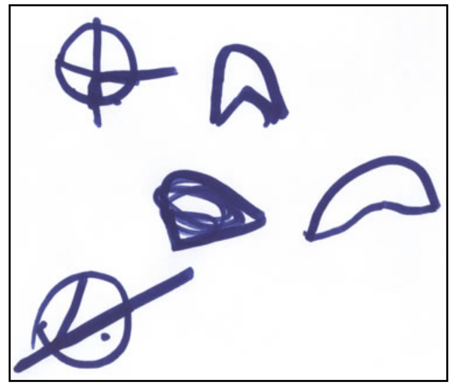
A part-whole perspective is only effective for substantiating equivalent fractions and comparing relatively close fractions if students identify and use relationships.

The most common approach for solving order and equivalence problems in this study was based on the part-whole perspective. Since fractions are often introduced and defined at the elementary level as parts of a whole, with representations depicting a whole divided into a specified number of pieces, this perspective might be anticipated. Even so, students who used a part-whole perspective did not always arrive at the correct solution when comparing and ordering fractions.

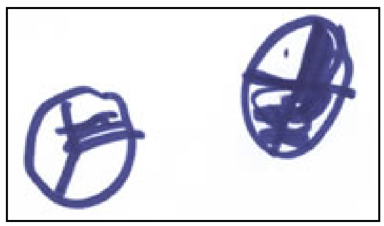
Almost all of the students were able to make simple drawings of fractions as parts of wholes, even though they had different understandings about the importance of equal-sized pieces. For some problems, the division into the number of parts did not require much accuracy such as comparing 3/16 and 5/8. But when the fractions were either equivalent or relatively close in size, a representation with multiple-sized pieces was inadequate. In these situations, making comparisons based on inaccurate representations resulted in incorrect answers. When students connected how to make the representations with other relationships, not only were their representations more accurate but they also mathematically supported their answers.

Mark, Danielle, and Elaine used a part-whole perspective to compare 2/3 and 3/4 on the pre-interview and each had different results (see Figure 6). Based on his drawing (Figure 6a), Mark decided the fractions were equal. Danielle's picture (Figure 6b) led her to believe that 2/3 was the larger fraction. Although Elaine's drawing (Figure 6c) was similar to Mark's, she decided that 3/4 was larger. She looked at the 3/4 and traced the inside angle that was missing. Elaine explained that 3/4 was larger "Because it is a right angle and the 1/3 has an obtuse angle, so it is leaving out more space."

a) Mark



b) Danielle



c) Elaine

Figure 6: Drawings of 2/3 and 3/4

When ordering the fractions 2/7, 7/15, and 6/10 on the post-interview, Todd and Krista made representations of the fractions on paper and determined the correct answer. Both sets of drawings were not accurate enough to prove 7/15 was smaller than 6/10, even though 6/10 looked larger in both pictures. Todd added a line at the half way point in Figure 7 and explained that 6/10 was "pass the half way point and this isn't [pointing at the fifteenths] because 7/15 - it would be 7/15 and a half of a fifteenth to be at the half way point." By relating his picture to a half, Todd confidently and competently answered the question. In comparison, Krista did not look for relationships to help her compare the fractions (see Figure 8). Instead she was confused by her picture, with the sections that lined up on the far right side of her drawing, that showed that "2/10 equals 1/7 I think. Uh-huh, these 2/10 equals 1/7."

Figure 7: Todd's Drawing of 7/15 and 6/10

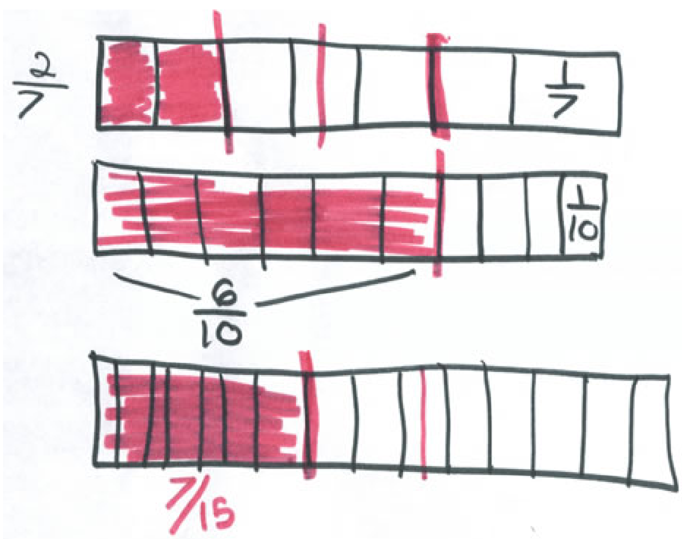


Figure 8: Krista's Drawing of 2/7, 7/15, and 6/10

During the pre-interview, six students attempted to compare 2/3 and 6/9 based on their part-whole perspective. Mark specifically mentioned, “I don’t know how to split a shape into ninths so I might have a little trouble with this problem.” Without a strategy for making the dividing lines in the circle, he ended up using a trial and error approach. Other students also struggled with how to divide a circle into ninths. After first using a circle, Christina switched to a rectangle and added lines from one end to another until she had the right number of pieces. These students were unable to support their answer. In contrast, Elisabeth and Marie used relationships to solve this problem and provided mathematical justification for their answer. Elisabeth used the fact 3 x 3 = 9 to divide a circle into ninths by first making thirds and then dividing each third into three pieces. Elisabeth extended the relationships between thirds and other fractions to prove that 8/12 and 12/18 were also equal to 2/3 and 6/9. Marie used a number line, which she called a “time line,” and wrote 0/0 on the left side and 3/3 and 9/9 on the right side. She divided the line into thirds by adding two hash marks and labeled 2/3 and 1/3 on the top and then added two hash marks within each third so that she divided each third into three more pieces. When I asked how she knew to make the ninths, she explained that, “3 x 3 = 9. Three little pieces would be one three, three more pieces would be two threes, and three more pieces would be three threes.” Unlike many students who struggled with making ninths, both Marie and Elisabeth used known relationships to plan how to draw their representations.

Some students started to use relationships in their drawings over the course of the unit because the relationships were brought out through classroom discussions or teacher guidance. For example, Mark tried to solve a variety of problems by drawing pictures, but he struggled with how to make equal-sized pieces using relationships. Ms. Bell approached him while he was trying to divide a rectangle into ninths. She suggested that, like the fraction kit, all of the dividing lines should to be in the same direction to make comparisons. She drew a rectangle, divided it into thirds, and asked Mark how he could make sixths. He first added a line perpendicular to the first lines, but Mrs. Bell asked if there was a way to cut lines all in the same direction. He divided each piece in half by adding a parallel line between the first cut lines. Mark then drew ninths by dividing a rectangle into thirds and then dividing each section into three more pieces. Unlike his previous attempts to draw fractions such as ninths, Mark thought about how to use relationships to draw equal-sized pieces and justify his answer.

This experience might have been a catalyst for Mark to think about how to draw fractions using known relationships such as between fourths and twelfths on the post-interview. For the last problem, Mark had to compare 2/8 and 3/12. These were the same fractions that he tried to compare on the mid-interview, but he could not divide a circle into twelve equal-sized pieces. Using skinny rectangles instead of circles, he divided both rectangles into half, then fourths. Then he divided each fourth piece in half on the first rectangle so he had eight pieces. He divided each fourth piece on the second rectangle into three pieces to make twelfths. After shading in the appropriate amounts, he drew two rectangles next his original drawing and called them 2/8 and 3/12. I proposed that the rectangles looked close, but it did not prove that they were equal. I asked if he could think of another reason why he thought they were the same. He responded, “Well, umm, this was first a fourth [pointing at 2/8] and this was also first a fourth [pointing at 3/12]. So basically I just shaded in a fourth on each of them.” His justification clearly explained the relationship between these two fractions that was also apparent in the way he drew them.

Although Mark learned how to use multiplicative relationships to help him, other students realized they needed equal-sized pieces, but were not sure what to do. These students either used trial and error or chose to solve the problem using a different perspective. Although Christina was not concerned about equal-sized parts on the pre-interview and decided that fractions such as 2/3 and 6/9 were equivalent based on a visual perception, this approach was not satisfactory to her by the mid-interview. She attempted to draw pictures to compare the fraction 10/12 and 5/6. Christina explained, “I thought it was really hard to draw sixths on a pie graph so I tried a bar. But then I realized that it was really hard for me to try to get it exactly right – to put the lines where they're supposed to be.” This evolution in her understanding of the part-whole perspective may have been due to activities in class where they talked about equal-sized pieces both in making the fraction kits and in drawing fourths. Christina moved away from trying to draw the fractions to using other relationships for comparing and ordering fractions.

Although students did understand fractions as part of a whole, this understanding was often not robust enough to compare and order fractions that were equivalent or close in size. To prove and justify fractions were equivalent based on a part-whole perspective, students needed to understand and use relationships. Using multiplicative relationships to construct the fraction brought out the equivalent relationships in fractions. Relationships were also essential when ordering fractions. Comparing representations to the whole or half was an effective way to evaluate the relative size of two fractions. These relationships required students to connect their view of fractions as parts of a whole to other understandings about fractions and whole numbers.

THEME THREE: USES OF PHYSICAL MODELS

Students use physical models in a variety of ways including: to determine an answer, to explain or justify answers, to show known relationships, and to remember relationships.

During all of the interviews, I had a variety of physical materials including paper available for students to use in solving problems. Students had a choice about whether to use any of the materials, which ones to use, and how to use the materials. Even though there were a variety of materials available, three physical models were used repeatedly: drawings, fraction strips kit, and paper folder. Each of these physical models is listed in Table 8 describing how it was utilized and how it connected to the students' perspectives about fractions. I describe how students used these physical models below.

Table 8: Use of Physical Models

Physical model	How physical model used	Perspective
Drawings	To compare fractions and find an answer	Part-whole
	To explain or justify an answer	
	To shown known relationships	Pieces
Fraction kit	To compare fractions and find an answer	Pieces
	To remember or demonstrate known relationships	Equivalence Within-fraction
Paper folding	To explain or justify an answer	Part-whole
	To compare fractions and find an answer	
No physical model	NA	Limited Unit Fraction Between-Fraction Transform

Students used drawings to figure out answers, to explain or justify answers, and to show known relationships. Todd used a drawing to compare 6/10 and 7/15. When Marie was comparing 2/3 and 6/9, she used the drawing to justify her answer. Joey used a picture to show the relationship between 3/16 and 5/8, carefully lining up 1/8 with 2/16. When students used drawings with a part-whole perspective, they began by drawing the whole and dividing it into the appropriate number of parts. With the exception of Marie who used a number line, students tended to favor area models with circles and rectangles. To make comparisons, relationships between the whole and the size of the pieces were very important.

When students had a pieces perspective, they only drew the pieces that were stated in the problem. This led to incomplete mathematical solutions.

When students used the fraction kit, they tended to use it both for figuring out answers and for remember or demonstrating known relationships. On the post-interview, several students used the fraction kit to put 1/2, 1/6, 2/3, 3/4 in order from smallest to largest and to determine that 3/12 and 2/8 were equal. During these solutions, students tended to exhibit a pieces approach because the fraction pieces could be manipulated without referring to the whole. Students also frequently used the fraction kit to remember and demonstrate known relationships. For example, students remembered relationships including 1/8 was equal to 2/16 and 1/4 was equal to 4/16. The development of these relationships using the fraction kit encouraged students to look for equivalent relationships when solving problems.

Students used paper folding less frequently than the other two physical models and only when they had a part-whole perspective of fractions. Usually students used this physical model to explain or justify an answer. For example, Allison folded paper to explain why 1/6 and 2/12 were equal to each other. Sometimes it was used to compare fractions such as when Arthur folded an index card to compare 2/3 and 3/4. Paper folding was not always a successful approach. Danielle decided to use paper folding to make fifths. After one fold, she said she had halves and after the second fold she said she had fourths. Danielle stated, “One more fold and I have fifths.” As she held up the paper, she said, “These are fifths!” and was surprised to see she had made eighths. Although the process of paper folding required the student to always start with the whole, the relationships involved in making different numbers of parts using paper folding were challenging.

There were additional physical models available during the interviews, but only one or two students occasionally used them. Marie tended to use the pattern blocks, especially on the post-interview. Arthur and Krista solved a problem using the interlocking cubes. A couple of students used the ruler, but mostly to measure and draw pictures. Although the teacher introduced two-colored counters as a set model in class and these materials were available on the interviews, student rarely used them. These physical models played such a minor role in solving problems that they were not included in the table.

Comparing across all three types of physical models, students used all of the materials to compare fractions and figure out answers. When planning how students might use different physical materials, expecting students to use the physical materials to figure out answers is logical. Students also used the physical materials as a way to explain or justify their answers. These students frequently had a different initial perspective to solve a problem, but they also had the flexibility to fall back on more concrete methods to prove their answer. Of course, sometimes students just wanted to show their answer in a second way and their use of the physical model did not actually provide a mathematical justification. Finally, students used the fraction strips to remember relationships. The ability of the students to draw upon these relationships indicated that students were developing a broader understanding of fractions through the physical models.

Physical models were not used for four of the perspectives: limited, unit fraction, between-fraction, and transform. Even so, students did refer occasionally to physical models when they solved problems using a unit fraction or transform perspective. Students with a unit fraction perspective frequently referred to physical models to explain the inverse relationship between the number of pieces and size of each piece. When I asked students to explain why the transform procedures worked, some students referred to or used physical models to explain the procedures in a concrete manner.

THEME FOUR: PERSPECTIVES AND QUANTITATIVE NOTION OF FRACTIONS

Each perspective highlights different relationships of fractions which are important in the development of a quantitative notion of fractions.

During instruction on rational numbers, we want children to develop "a quantitative notion of rational number" (Post et al., 1986, p. 40). For students to be able to use number sense and choose efficient strategies for comparing and ordering fraction, they need to understand what information is relevant. Table 9 summarizes the aspects of a quantitative notion of fractions which are apparent in the reasoning of each perspective.

The limited and pieces perspectives do not necessarily help children develop a quantitative notion of fractions, but illuminate which relationships students are not aware of. For example a child who uses whole number strategies for ordering fractions does not understand how fractions are different from whole numbers. A child who uses a pieces perspective does not realize the importance of the whole in determining the size of the pieces. The transform perspective is not included in Table 9. When students are using procedures to compare and order fractions, they are usually not attending to relative size. If students examine their work derived from transformations to think about whether an answer makes sense or to prove an answer using a substantive argument, they rely on relationships that are part of other perspectives.

Table 9: Perspectives and Quantitative Notion of Fractions

Perspective	Description	Quantitative Notion
Limited	Lack of understanding about fractions	Limited
Pieces	Focus on fractions as pieces independent of the whole	Limited, absolute size
Part-Whole	Focus on fractions as parts of a whole	Relative size, size of the whole
Unit Fraction	Focus on unit fractions	Relative size
Within-Fraction	Focus on the relationship between the numerator and denominator	Relationship between numerator and denominator
Between-Fraction	Focus on the relationship between numerators and/or denominators	Multiple ways to represent a fraction
Equivalence	Focus on the relationship between equivalent fractions	Multiple ways to represent a fraction

When students attended to the different relationships important in developing a quantitative notion of fractions, they built on strategies within one perspective to quantify fractional relationships using other perspectives. Although not part of the within-fraction perspective, some students extended the benchmark strategy to quantify the fractional amount that a fraction was over or under 1/2.

For example, one student was comparing 3/5 and 4/6 when she explained, “Four-sixths would be, would be 1/6 away from a half which is 3/6. So it's 1/6 away from a half. Three-fifths is a half away from a half. I mean a half of fifth, a tenth away from a half.” Not only was she able to explain her answer, she actually quantified how much more each of these fractions was from a half.

Marie and Elisabeth relied on their growing understandings about the relative size of fractions to approximate the size of 2/7 during the post-interview. Marie claimed that 2/7 was close to a third explaining, “Two-sevenths is close to a third because three and a half sevenths would be one-half, and so it's like… a half is a sixth more than a third.” Elisabeth used other relationships to compare 2/7 to 1/4: “I know that three and a half is half of seven. And two is half of four. And this is just one half away from four, so I know that this is kind of - not too close to 1/4 - but I say it's about as close to 1/4.” This flexibility to use different attributes of fractions to reason about the relative size demonstrates the power of developing a quantitative notion for fractions.

Do students use a specific perspective because they have a specific quantitative notion of fractions? Or do they develop each quantitative notion of fraction because they focus on relationships from a specific perspective? Either of these explanations may be valid, but it is also possible that the development of a quantitative notion of fractions and perspectives are interdependent and recursive. Students use a certain perspective because they have a certain quantitative notion of fractions, and through the process of using the perspective, they are developing a stronger understanding of the underlying relationships that are important to understanding relative size of fractions.

THEME FIVE: FLUENCY FROM UNDERSTANDING RELATIONSHIPS

Students who have stronger understanding of relationships in fractions can attend to the most salient features for comparing and ordering fractions, move fluently between different perspectives, and use multiple approaches to justify and explain answers.

When students understood more relationships in fractions, they tended to move between different perspectives to solve problems. They developed fluency with solving problems that allowed them to solve problems and justify their answers very efficiently and effectively. When asked if there was another way to solve a problem, these students used a strategy from a different perspective. Students with an understanding of fewer relationships did not have the same flexibility to solve problems. They often tried to solve a problem one way and were not able to use another strategy. Although all students fell somewhere on this continuum in the number and types of relationships they understood and that guided them in solving problems, I included examples from different ends of the continuum to illustrate these differences.

Fewer relationships: Bobby and Mark

Bobby and Mark were fourth graders who had fewer relationships to draw upon and were less likely to solve problems using a variety of perspectives. Bobby was strongly influenced by his pieces and unit fraction perspective, while Mark's part-whole perspective dominated his approach to solving problems.

Bobby often solved problems using a pieces perspective. For example, he used the fraction strips to compare 2/8 and 3/12. When asked for a second way of solving the problem, he drew a picture of the fractions that matched the representation with the fraction strips. When I asked him questions that could not be solved using the fraction kit, he switched to the unit fraction perspective. He decided that 3/5 and 4/6 were equal because the fifths were larger and than the sixths, and there were more sixths. He used similar reasoning to compare 7/15 and 6/10. He was unable to think of another way to compare these fractions, even though the teacher emphasized using multiple solution strategies in class. The only other perspective he used on the mid- and post-interviews was equivalence, and two of the three equivalent relationships were based on the fraction strips. Bobby's ability to solve problems in multiple ways was limited by the number of relationships he understood and could apply.

Mark had more ways to solve problems than Bobby, but the part-whole perspective dominated how he solved problems. Even when he could explain an answer without the pictorial representation, he drew pictures. He frequently started with circles and added lines to make a certain number of pieces, but was unable to plan ahead or use relationships to make his drawings. When he was uncertain about how to divide a shape, he expressed concern and even struggled with solving a problem. At times, he recognized relationships that occurred when he divided all of the pieces in half, but he was not able to use or extend this relationship to make specific fractions such as twelfths. Sometimes he compared fractions that were close in size and his pictures were not useful for figuring out or explaining his answers. His drawings changed over time as he started to use relationships to make his drawings and comparisons. By the end of the unit, Mark started to think about how to draw different fractions based on relationships that he already knew. The use of the relationships in his drawings allowed the part-whole perspective to be a more powerful approach to compare and order fractions, but he was unable to connect these relationships to other perspectives.

More relationships: Marie and Christina

In comparison to Bobby and Mark, Christina and Marie had more flexibility to move between different perspectives and solve problems more easily. Over the course of the unit, Christina became aware of more relationships and used them to help her with solving different types of problems. From the beginning of the unit, Marie had more flexibility, but she connected the procedures from the transform perspective to other more concrete perspectives to explain why these procedures worked.

Based on her understanding of fractions from her informal knowledge and limited prior experience, Christina solved some order and equivalence problems prior to instruction. Christina originally tried to solve problems using a part-whole perspective, but she did not use relationships to represent the different fractions. By the mid-interview, she realized that the part-whole representation required equal-sized pieces. When she could not make her drawings with equal-sized pieces, she changed her approach for solving problems. This pushed Christina to find relationships and use her understanding of these relationships to justify her answers. During the post-interview, Christina solved complex problems that included a variety of fractions, many that she had not worked with previously. For example, she used benchmarks to order 2/7, 7/15, and 6/10 and to compare 3/5 and 4/6. She did not need physical models because she used relationships that she was building between different fractions including benchmarks to solve these problems. One of the striking differences over time with was how important the use of relationships became in solving problems.

Although Christina developed relationships over the course of the unit, Marie had many relationships to use even on the pre-interview. Marie successfully solved all of the problems asked in the three interviews using a variety of methods and perspectives. By the end of the unit Marie chose efficient and logical approaches for the specific numbers involved, which allowed her to solve comparing and ordering fractions with ease.

One aspect of Marie's developing fluency can be observed in her explanation of the procedures for proving equivalent relationships and generating new equivalent fractions based on the transform perspective. Even though Marie correctly used these procedures over all three interviews, her explanation for why it worked was limited at the beginning. At the end of the unit, Marie proved that 4/10 and 2/5 were the same amount by dividing and multiplying by 2/2. I mentioned that some of her classmates did not understand this strategy and asked if she could think of another way to explain why these fractions were equivalent. Marie drew a rectangle (see Figure 9), divided it into five sections, and said, "If I have fifths, I have fifths, right? And I only have these two." Marie drew a dot above each of the first two sections and shaded them in. Then she added three unshaded dots above the other sections. She explained, "If they're shaded it in, it also means I have them. And if I split it in half, each fifth in half, if I split all the fifths in half." Marie added lines parallel to her original lines to show cutting each one in half so she now had ten sections. Marie continued, "Then I'd have one, two, three, four," shading in one section at a time as she counted, "So 4 is my numerator. Four out of 10 or 4/10." Using a part-whole perspective, Marie clearly explained and demonstrated the connection between the procedure and a physical representation. Marie followed a similar process to demonstrate the relationship between 2/5 and 6/15. In addition to being able to use the rule for generating equivalent fractions correctly, she understood why the equivalent relationship was maintained.

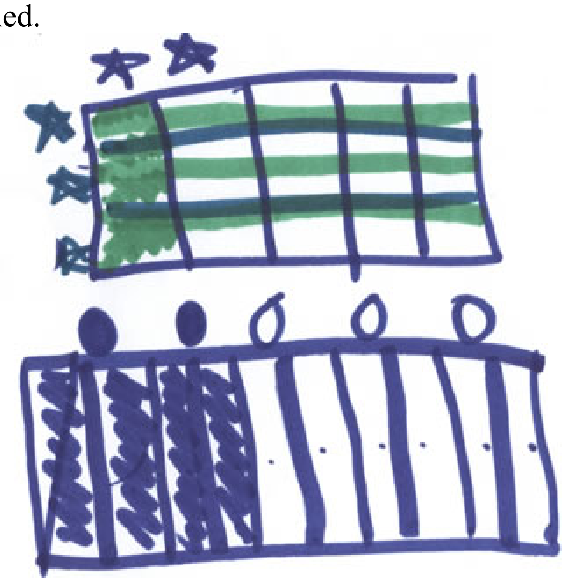


Figure 9: Marie's Drawing of 2/5, 4/10, and 6/15

A second aspect of Marie's fluency can be seen in how she solved problems where the fractions were not equivalent. Although Marie knew how to use common denominators to find equivalent fractions, she chose not to use these procedures for comparing and ordering fractions. For example, she used benchmark fractions to compare 3/5 and 4/6, even though they could be compared using a common denominator. She used her understanding of fractions and relationships between fractions to solve ordering and comparing problems. By the end of the unit, she recognized relationships that she could use to compare specific fractions that were efficient and valid.

Although Marie approached many problems using transform procedures, she had the flexibility to move from symbols to representations to verify or justify her answers to others. Marie used relationships in fractions and in numbers to solve comparing and ordering fraction problems from both symbolic and representational aspects. Although Marie had an understanding of fraction concepts that allowed her to work at the symbolic level, her understanding was also grounded in representations from her previous experiences. By watching Marie and examining her work, it became apparent that Marie had a fluency that allowed her to move between different perspectives and use different methods for comparing and ordering fractions.

SUMMARY

These themes demonstrate the importance of number sense, especially when comparing and ordering fractions. For children to develop number sense with fractions, they have to become aware of fractional relationships. For example, students should examine and describe the relationships within manipulatives, observe and use relationships in their drawings, and investigate and extend numerical relationships. Making connections between various types of representations are part of developing an understanding of fractions. By identifying and building upon relationships in fractions, students will develop number sense and will become fluent in solving problems related to order and equivalence.

Chapter 6: Conclusions and Implications

OVERVIEW OF THE STUDY

This study examined students' use of physical models and their developing understanding of fractions as they solved order and equivalence problems. Thirteen students in a combined third, fourth and fifth grade class participated in this study. They learned mathematics in a constructivist class where the teacher based the development and implementation of the unit on a Cognitively Guided Instruction (CGI) approach for teaching mathematics. I videotaped the class daily during the fraction unit and interviewed all of the students prior to instruction and immediately following the unit. Eight students also participated in interviews approximately midway through the unit.

SUMMARY OF FINDINGS

To summarize the findings, I return to the research questions posed in chapter one. The perspectives described in chapter four and the themes from chapter five answer each of the research questions.

To what relationships do elementary students attend and utilize when comparing and ordering fractions? Students used a variety of relationships when they were comparing and ordering fractions. These relationships provided the basis for the perspectives. When students did not pay attention to relationships or relied on invalid relationships, they had a limited perspective. For example, students with a pieces perspective identified and used relationships between different pieces, but they did not refer to the whole. The part-whole perspective required that students paid attention to the relationship between the parts and the whole. Even so, students did not always understand or address some important part-whole relationships such as equal-sized pieces and equal-size wholes. The unit fraction perspective was implicitly based on the part-whole perspective; however, the primary relationships students focused on were between unit fractions. Students considered composite fractions as iterated unit fractions. When students had a within-fraction perspective, they used relationships between the numerator and denominator of a fraction. Many students extended the relationships between the numerator and denominator to judge the proximity of a fraction to a benchmark. Students with a between-fraction perspective found both additive and multiplicative relationships across numerators and denominators. Students who identified equivalent relationships and extended those relationships to other fractions had an equivalence perspective. Students with a transform perspective used number relationships to determine common denominators and generate equivalent fractions. With the exception of the limited and transform perspectives, the relationships embedded in the perspectives directly connected to students' development of a quantitative notion of fractions.

How are physical models utilized and extended by elementary students for comparing and ordering fractions in a constructivist mathematics class? Students used physical models in several different ways. One approach was students represented the fractions in the problems using either manipulatives or drawings and compared the amounts to figure out the answer. When students used the manipulatives, they often worked with the individual fraction pieces and did not refer to the whole. In using drawings, most students began with a whole and divided it into the number of pieces indicated by the denominator. If students answered a question without using physical models, they sometimes explained or justified their answer using manipulatives or a drawing. Finally, students used the physical models to help them remember equivalent fractions. After students explained how they used a specific fact to help solve a problem, I asked how they knew the fact. They referred to the physical models, usually the fraction strip kits. Students built upon these number facts by figuring out other equivalent relationships, such as deciding that 6/12 equaled 1/2 although there were not any twelfths in the kit at the time.

How do children's approaches to solving order and equivalence fraction problems and the use of physical models support the development of number relationships? When students solved order and equivalence problems, they used and developed number relationships at the same time. To make drawings to compare and order fractions, students used multiplicative number relationships to make partitions, and they later used these relationships to justify their answers. Students who understood the inverse relationship between the numerator and the size of the fraction piece used the unit fraction perspective to solve problems. Students used both additive and multiplicative relationships when they compared fractions based on either a within-fraction or between-fraction perspective. Though many additive relationships are invalid, the students in this study used additive relationships and maintained the ratio relationship between numerators and denominators to identify equivalent fractions. Students learned a limited number of equivalent fractions by working with physical models; and they made generalizations to generate other equivalent relationships not represented by the physical models. Students who moved between different perspectives and chose efficient strategies were more successful in solving problems and in explaining their answers, thereby demonstrating the importance of using number relationships to understand fraction concepts.

SIGNIFICANCE OF FINDINGS

Perspectives

Smith's (1995) perspectives contributed a framework for categorizing the strategies students used for comparing and ordering fractions. His perspectives also provided the framework for this study. After describing each perspective identified by Smith, I compare the related perspectives that emerged from my data to Smith's work and other research. I also describe the significance of the three perspectives that did not fit within Smith's framework. I present some possible reasons for differences at the end of this section.

Parts Perspective

My results indicated two approaches for solving order and equivalence problems embedded in Smith's (1995) parts perspective. The less sophisticated approach that emerged from my data was the pieces perspective. Students only focused on pieces and did not refer to the whole. This is supported by Armstrong and Larson (1995) who found students using a direct comparison strategy were concerned about the parts and not the relationship between parts and wholes. The use of the fraction strip kits during instruction further supported this perspective and was similar to Ball's (1993) observations.

The more sophisticated approach was the part-whole perspective related to Smith's description of the parts perspective. Although Smith required equal-sized parts, many students in my study did not use equal-sized parts. When students decided to make equal-sized pieces, they were successful only if they drew upon other mathematics facts to help them. Smith also included mental models in his description of the perspectives; however I interpreted the part-whole perspective as being observable through students' drawings or use of manipulatives.

Components Perspective

Smith (1995) defined the components perspective based on students' use of natural number relationships either within or across numerators and denominators. Like Smith, I found that students used both within and between fraction relationships to make judgments for comparing and ordering fractions, but I divided these into two perspectives because students focused on different relationships with each of these perspectives.

Students who used a within-fraction perspective recognized that the relationship between the numerator and denominator was important. Since students had to use the relationship between the numerator and denominator to make comparisons to reference points, I included the benchmark strategies in this perspective. This was a key aspect of developing a quantitative notion of fractions (Post et al., 1986). Post et al. (1985) demonstrated how a student progressed from using physical models to making generalizations based on the relationship between the numerator and denominator.

Students with a between-fraction perspective used relationships across numerators and across denominators to make comparisons. Research has shown that student initially use doubling or halving strategies for generating equivalent fractions (Brinker, 1997; Smith, 1995; Streefland, 1993). Similar to Smith, I found that students used both additive and multiplicative reasoning for within-fraction and between-fraction relationships, which is supported by previous findings (Behr et al., 1984; D'Ambrosio & Mewborn, 1994; Kaput, 1994; Post et al., 1986; Wearne-Hiebert & Hiebert, 1983).

Reference Point Perspective

As opposed to students who used a components perspective, Smith (1995) claimed students focused on the fractions instead of natural numbers in the fractions when they had a reference point perspective. Using reference points or benchmarks strategies is documented in the literature (Behr et al., 1984; Post et al., 1986; Smith, 1995; Zeman, 1991).

In contrast to Smith (1995) who identified a separate reference point perspective, I included benchmark strategies in the within-fraction perspective because students considered the whole number relationships between the numerator and denominator before they could make a comparison. When the students in my study compared fractions to benchmarks, they divided the denominator in half and often referenced the half-way point. For example, half of five is two and a half, so one half equals two and a half fifths. Although none of the elementary students in Smith's study used these strategies to make comparisons, third, fourth, and fifth graders in my study did use benchmarks. I believe that this difference was due to the nature of instruction in my study where the teacher included activities to develop benchmark strategies.

Transform Perspective

The transform perspective was the same in both of our frameworks, although Smith (1995) observed it more frequently. This was probably due to the fact that his students were fifth grade or higher, in a traditional mathematics class, and the analysis of the textbook confirmed that transform strategies were explicitly taught. Since the students in my study were in a Cognitively Guided Instruction class, they were not taught specific procedures for comparing and ordering fractions. Students in my study learned transform procedures by watching more advanced classmates or from their parents.

Additional Perspectives

I identified three additional perspectives that were not included in Smith's (1995) framework: *limited, unit fraction,* and *equivalence*. The *limited perspective* demonstrated students did not have valid relationships to help them solve order and equivalence problems. Previous studies identified strategies included in this perspective such as making comparisons based on whole number relationships (Ball, 1993; Behr et al., 1984; Moss & Case, 1999; Streefland, 1993; Vance, 1986) and using additive relationships instead of multiplicative relationships (Behr et al., 1984; Post et al., 1986; Smith, 1995; Wearne-Hiebert & Hiebert, 1983). Additive relationships only fit in the limited perspective when students used them without maintaining the ratio relationship between the numerator and denominator in the fraction, such as when they added the same number to the numerator and denominator. Students in my study also invented their own rules for comparing fractions. Smith identified strategies that were invalid or limited, but he categorized them into his parts and components perspectives. I chose to keep them as a separate perspective because students demonstrated they needed to learn new relationships to help them solve order and equivalence problems.

Smith (1995) described several strategies in both the parts and components perspectives that I believe belong in their own category, which I called the *unit fraction perspective*. Students used the inverse relationship between the number in the denominator and the size of the fraction to order problems with the same numerator (Behr et al., 1984; Post et al., 1985; Smith, 1995). This understanding was based on the part-whole relationship: each piece is smaller as there are more partitions. Another strategy students used was determining that the greater fraction had the smaller denominator (so each piece was larger) and the larger numerator (so there were more pieces). When these conditions were not met, some students used qualitative reasoning to decide whether the size of the piece or the number of pieces had a greater effect (Behr et al., 1992; Smith, 1995). As students considered both the numerator and denominator, they treated composite fractions as iterated unit fractions. Behr et al. (1983) claims this understanding helps “children develop a stronger quantitative notion of rational numbers” (p. 123). The Fraction Project hypothesized that students needed to develop this understanding of fractions as an iterable unit fraction so they planned instruction that focused on this concept (D'Ambrosio & Mewborn, 1994; Tzur, 1999). By teaching fractions using the Stick microworld, Tzur (1999) found that students understood fractions “as a single quantity,” which he expressed demonstrated a stronger understanding than the “parts of a whole” meaning of fractions. My study demonstrated that students can construct their understanding of composite fractions based on iterating unit fractions.

Smith’s (1995) study did not address the *equivalence perspective* in his framework. He mentioned the recalled fact strategy in his dissertation, but it occurred so infrequently that he did not include it in any of the perspectives (Smith, 1990). Students in my study remembered facts, often due to their work with physical models. As students moved beyond the equivalent fractions in the physical models, they tended to use the relationship that resulted from splitting a unit fraction into two equivalent pieces, such as 1/5 equaled 2/10. Students built on these relationships by incrementing, so if 1/5 equaled 2/10 then 2/5 equaled 4/10 and 3/5 equaled 6/10. Even though using equivalent relationships was a powerful approach for the students in my study, I have not found similar strategies documented in the literature.

Summary of Perspectives

This section demonstrated both the similarities and difference between the perspectives that Smith (1995) and I identified. Some of the differences are due to how we categorized students’ general approaches. For example, he clustered within-fraction and between-fraction approaches in the components perspective, whereas I kept these as two separate perspectives. Many of the variations between the results of our studies were due to the differences in the settings and participants. First, the students in my study were younger, so they were still developing their understanding of fraction concepts. Secondly, since the students in my study were in a constructivist class, they were learning mathematics in a very different manner from students in a traditional class. Lastly, through the interaction of grade levels and the learning environment, students used physical models including manipulatives and drawings as an integral part of how they were learning about fractions.

Themes

The themes provided connections between the perspectives and how children solved problems. As indicated in the descriptions, the piece and part-whole perspectives usually included using physical models. Since the students constructed the fraction strip kits and used them for mathematics activities, this physical model played a very important role in the development of relationships that students used. Given that students had to record their answers in their mathematics journals, pictures were another important model for solving order and equivalence problems. Students used physical models for more than just figuring out right answers. The themes that emerged from the data were also connected to children's developing understanding of fractions. Using various relationships and moving between perspectives permitted students to identify multiple and efficient approaches for solving problems.

Theme One: Area and/or linear pre-partitioned physical models can positively impact children's development of number relationships for comparing and ordering fractions when students examine the relationships between wholes and different parts and are able to extend these relationships to fractions not included in the physical models.

The most significant finding related to theme one was that students did not necessarily interpret pre-partitioned physical models as connected to a part-whole representation for fractions, even though researchers intend for these materials to help children develop this understanding of fractions (Behr et al., 1984; Cramer et al., 2002; Post et al., 1985). Instead, some students in my study associated certain-sized pieces as specific fractions and identified relationships between different-sized pieces, but they did not connect the size of the piece to the size of the whole. Although mathematics concepts are apparent to individuals with an understanding of the underlying concepts, researchers have expressed concerns that the concepts are not apparent in the manipulatives for individuals who are still learning the concept (Behr et al., 1983; Gravemeijer, 1997; Thompson & Lambdin, 1994). This finding demonstrated that although experts may intend for students to understand certain relationships using manipulatives, these are not the relationships that students automatically address. Ball (1993) previously observed this when students started to associate a visual representation of a fraction with a certain shape. For example, students started to draw a quarter of a circle to always represent one-fourth. Ball speculated that this focus on the part without the whole was an artifact of the representations used in class. This study provided more evidence that certain physical models allowed students to develop a visual representation of fractions without a connection to the whole.

Theme 2: A part-whole perspective is only effective for substantiating equivalent fractions and comparing relatively close fractions if students identify and use relationships.

This study provided evidence for Pothier and Sawada's (1983) hypothetical fifth level of partitioning which they called composition. I submit that this level is necessary if students want to use partitioning to prove fractions are equivalent or to compare fractions that are close in size. At this fifth level, students use multiplicative relationships to create equal-sized parts for fractions with composite numbers in the denominator. For example, students partitioned a whole into ninths by making thirds and dividing each third into three pieces. Steffe and Olive (1991) showed students can find equivalent relationships for fractions by adding or removing partitions. Adding and removing partitions required children to think about the multiplicative relationships inherent in equivalent fractions. However, students who did not recognize the importance of equal-sized pieces did not plan how to use relationships to make partitions. Students who wanted to make equal-sized pieces but did not know how to use relationships had to rely on a different perspective for solving problems.

Theme 3: Students use physical models in a variety of ways including: to determine an answer, to explain or justify answers, to show known relationships, and to remember relationships.

Physical models were used in this study both as "tools to get an answer" and as "tools for thought" (Kent & Gravemeijer, 2001). The findings confirmed students sometimes used the physical models as "tools to get an answer" when problems could be solved using the materials. This is similar to other studies that demonstrated while students made direct comparisons, they used an "end-to-end" strategy with manipulatives or based their answers on perceptual cues and did not examine other relevant relationships (Armstrong & Larson, 1995; Brinker, 1997; Kamii & Clark, 1995; Streefland, 1993). Sometimes students solved problems without using physical materials, but referred to or actually used the materials in explaining and justifying their answers. On the other hand, when problems could not be solved using physical materials, the materials became “tools for thought” and supported development of number relationships. Kent and Gravemeijer (2001) recommended presenting problems with numbers not included in manipulatives so that students had to build new relationships. This was most apparent in my study when students generated equivalent relationships not represented in the physical models to solve problems.

Theme 4: Each perspective highlights different relationships of fractions which are important in the development of a quantitative notion of fractions.

Post et al. (1986) conjectured that developing a quantitative notion of fractions was interlocked with learning to how to compare and order fractions. Students used specific relationships that they understood to make comparisons; and as they made comparisons, students understood new relationships. This study verified interconnections between the children’s developing understanding of fractions and how children solved problems. When students had a part-whole perspective, they used and learned about relative size and the importance of the whole. The unit fraction perspective depended on understanding relative size, and students developed their understanding of relative size when they relied on the unit fraction perspective. Students were aware of the importance of the relationships between the numerator and denominator when they used a within-fraction perspective. When they solved problems using a within-fraction perspective, they learned more about the relationships between numerators and denominators. As students understood there were many ways to name the same fraction, they used between-fraction and equivalence relationships to generate more fractions. Students' ability to move between the different perspectives indicated a strong understanding of fractions, which is further addressed in the last theme.

Theme 5: Students who have stronger understanding of relationships in fractions can attend to the most salient features for comparing and ordering fractions, move fluently between different perspectives, and use multiple approaches to justify and explain answers.

Students in this study who identified and applied multiple relationships learned how to fluently solve a variety of order and equivalence problems. This finding related to Smith's (1995) discovery that competent students moved between different perspectives to solve problems. When children had multiple strategies for solving a problem, they chose specific strategies based on the type of problem and the numbers in the problem (Post et al., 1986; Smith, 1995). Results of these studies support the instructional approach introduced by RNP where students made transformations within and between representations including manipulatives, pictures, word, symbols and real-life situations (Behr et al., 1983; Cramer et al., 2002; Post et al., 1982; Post et al., 1985). In the RNP studies, students compared and discussed differences between various representations as they made transformations. Some students who used procedures to solve problems in Smith's (1995) and my study had the flexibility to use other perspectives to explain and justify their answers. I postulate that this ability to choose strategies and to move between perspectives to solve problems and justify answers is based on relationships that children can use. By creating many connections based on the relationships, students were developing a stronger understanding of fractions which they used to solve problems efficiently.

Limitations on Findings

There are several limitations to the findings of this study. The perspectives that I identified were based on a small group of students' approaches to solving problems in an unusual classroom setting where they were expected to construct their own strategies and justify their answers. The descriptions of these perspectives are preliminary, and these descriptions could be refined through a collaborative research study. The inter-rater reliability of coding data could improve as these perspectives are more clearly delineated. Finally, the transferability of these perspectives to students receiving instruction in traditional mathematics classes may be limited. Even with these limitations, the findings from this study have implications for both how children learn fraction concepts and how teachers can support their development.

IMPLICATIONS OF FINDINGS

Many of the findings in this study have implications for teaching and learning. Some of the concerns about what children learned from using pre-partitioned physical models or making their own drawings can be addressed through instruction. The perspectives also provide a framework for thinking about how children solve fraction problems in the classroom. A teacher can encourage children to develop important relationships by choosing activities carefully and asking questions that push students to examine and use relationships.

Physical Models

Whether students are using manipulatives or drawing pictures, students can make important connections if relationships are addressed. For example, one of my findings demonstrated that the discrete physical models impacted students understanding of fractions as part of a whole. Teachers or researchers cannot assume that students are connecting the fraction piece to the whole, so the relationships between the physical models and the whole must be made explicit. During instruction this may require asking questions such as “What is the whole?” and “Why is it important?” Also, questions that require students to consider the relative and absolute size of the whole (Post et al., 1986) can help students understand the importance of the whole. This can be accomplished by changing the size of the whole with prefabricated models such as pattern blocks or Cuisenaire rods or constructing models using different-sized wholes. In addition, asking questions similar to “How can someone eat 1/4 of a pizza and get more pizza than someone who ate 1/2 of a pizza?” requires students to consider the size of the whole.

Another important issue arises when students are drawing or creating their own representations for fractions. Since students have to reach the composition level (Y. Pothier & Sawada, 1983) with partitioning to prove fractions are equivalent or to compare close fractions, teachers need to bring out relationships. Students can learn about the relationships by discussion their strategies for partitioning wholes. A teacher can guide a student by asking questions such as “How can you use thirds to help you make sixths? Ninths?” or “What relationships can you use to help you draw twelfths?” These relationships should also be examined when children use manipulatives.

Perspectives

Based on the analysis of the data, children’s ability to solve order and equivalence problems was both helped and hindered by different perspectives. A students’ use of a certain perspective to solve and justify their answer demonstrated the benefits of relying on that perspective. However, as students tried a certain perspective and ended up with the wrong answer or could not solve the problem, the limitations of the perspective emerged. Some of the perspectives indicated that students were developing useful relationships for solving problems, whereas other perspectives indicated that children were developing limited relationships. Regardless of the perspective students used to solve problems, there are implications for teaching and furthering children’s learning. Table 10 summarizes the benefits, limitation, indications, and implications for each of the perspectives.

Table 10: Benefits, Limitations, Indications, and Implications of Perspectives

Perspectives	Benefits	Limitations	Indications and Implications for teaching
Limited	None	Leads to incorrect solutions OR Correct answers for invalid reasons	Indicates a lack of a "quantitative notion of fractions" Develop fraction concepts
Pieces	Ability to solve some problems, especially with manipulatives	Lack of connection to the whole	Indicates a limited understanding of fractions Connect the size of the piece to the whole
Part-Whole	Useful model for understanding fractions Ability to recreate a fraction	Possible to ignore key aspects of fractions like equal-sized pieces Without the use of relationships, difficulty to create certain fractions	Indicates a basic understanding of fractions Connect relationships between known facts and fractions
Unit Fraction	Able to compare relative size of unit fractions and some non-unit fractions	Difficult to use for all non-unit fractions Does not allow comparison of equivalent fractions	Indicates understanding of unit fractions Develop additional perspectives
Within-Fraction	Use of relationships (or approximate relationships) between the numerator and denominator to examine relative size of fractions	Requires flexibility to look for exact and approximate multiplicative relationships	Indicates an understanding of importance of relationships between numerator and denominator in fractions Encourage students to use these relationships to determine relative size

Perspectives	Benefits	Limitations	Indications and Implications for teaching
Between-Fraction	Use of number patterns across numerators and across denominators Awareness of ratio relationships	Some relationships across numerators and across denominators are difficult to determine or observe Relationship between numerator and denominator in understanding relative size of fractions may be obscured	Indicates some understanding of ratio relationships with fractions Encourage students to examine within-fraction relationships
Equivalence	Use of equivalent relationships Flexibility to use an equivalent fraction when it facilitates solving problems	Dependent on how the known equivalent relationships or ones that can be developed by students	Indicates understanding of many ways to express the same fraction Encourage students to look for equivalent fractions and use patterns to build more equivalent fractions
Transform	Ability to quickly and efficiently solve problems	Must remember rules Focus on procedures, not understanding	Being able to do the procedure does not indicate understanding of underlying fraction concepts Connect symbolic rules with other perspectives.

In chapter 4, I stated that the perspectives were generally listed in order of sophistication. The information in Table 10 supports this claim. The differences between the limited, pieces, and part-whole perspectives and the within-fraction, between-fraction and equivalence perspectives are most apparent. The benefits for the first three perspectives are not connected to developing key relationships for fractions and limitations may prevent students from using accurate mathematical reasoning to solve problems. When students used the first three perspectives, it indicated that they were still developing an understanding about fractions and their ability to compare and order fractions was restricted. The unit fraction perspective fits between the first three perspectives and the within-fraction, between-fraction, and equivalence perspectives. The unit fraction perspective was a restrictive perspective in some ways because it was useful for a limited number of fractions. In other ways, this was a flexible perspective because students used quantitative and qualitative reasoning to make judgments about relative size.

Students who used within-fraction, between-fraction, and equivalence perspectives had a stronger foundation for solving order and equivalence problems. Students understood some of the quantitative notions of fractions which they used to make comparisons. Though students had to identify certain relationships for these perspectives, they benefited by using and extending relationships. When students use these perspectives, instruction needs to encourage further development of relationships.

The transform perspective is an efficient approach for comparing and ordering fractions. Despite being listed last, teachers should not assume that the transform perspective indicates the highest level of a students' understanding. Students at this level may have an instrumental understanding (Skemp, 1978) of how to solve order and equivalence problems and may provide an analytical argument (Kent & Gravemeijer, 2001) describing the steps that they followed. These students are not at the most sophisticated level when they use the transform perspective. On the other hand, students working at the highest level of the transform perspective will demonstrate a relational understanding (Skemp, 1978) of transformations and will support their answers using substantial arguments (Kent & Gravemeijer, 2001).

One of the primary goals for fraction units should be to help children acquire multiple perspectives for comparing and ordering fractions by focusing on developing relationships. The classroom structure must encourage the development of conceptual understanding; and the activities during mathematics class must encourage students to search for and describe relationships. Kazemi and Stipek (2001) identified aspects of a high quality inquiry based mathematics class that developed children's conceptual understanding, which are also applicable to teaching fractions. For example, students solved problems using multiple strategies, made comparisons between different strategies, used different answers as opportunities for exploration and discussion, and provided substantial arguments that went beyond explaining steps to include justifying answers. By having students engage in activities where they observe, generalize and extend relationships, instruction can help students develop numbers sense for fractions and fluency with solving fraction problems.

RECOMMENDATIONS FOR FUTURE RESEARCH

There is more to learn about children's understanding of fraction concepts and how the use of physical models impacts their ability to solve order and equivalence problems. The perspectives provide a framework for organizing the types of relationships that were important to elementary children in this study. Future research needs to scrutinize and refine these perspectives by collecting data in multiple classroom settings with diverse populations, including adults. The role of different physical models in these perspectives should be studied in more detail. A major question is how does instruction impact the development of perspectives? Examining the development of a specific perspective by choosing activities that focus class discussions on certain relationships, research can study how students apply the relationships to other types of problem solving situations. Comparing how students solve order and equivalence problems in traditional classes versus inquiry-based classes may provide evidence of how different approaches to instruction impact the perspectives that students develop and use. Researchers can also focus on the development of specific perspectives. For example, they can investigate the factors that impact which relationships students attend to by carefully selecting numbers that can be solved using both within and between-fraction perspectives.

SUMMARY

This study examined children's use of relationships and physical models as they solved order and equivalence problems. I conducted this study in a third, fourth and fifth multi-grade class where the teacher used a Cognitively Guided Instruction approach for teaching mathematics. This unique setting was ideal for this study because students used relationships that were important to them and used physical models in ways that made sense to them. They explained their thinking about how they solved problems as they used the physical models, wrote symbolically, and justified their answers. In addition to daily observations throughout the fraction unit, I conducted two or three individual clinical interviews before, during and after the unit with each of the 13 students in the class.

My findings are interconnected with the relationships that students identified and used to solve order and equivalence problems. Smith (1995) described four general approaches or perspectives that students in his study used to compare and order fractions. However examining the relationships that students used, I found eight perspectives that overlapped in some ways, but in other ways were distinct from, Smith's descriptions. Further research could refine these perspectives and evaluate how physical models influence children's approaches for comparing and ordering fractions. In my study, the relationships that students attended to in the physical models shaped their developing understanding of fraction concepts. Relationships developed from using physical models and symbolic representations of fractions were important to students in their process of understanding fractions and learning to solve fraction problems efficiently. Students identify and use relationships in physical models and symbolic notation that are significant to them. To develop children's understanding of fraction concepts, teachers must observe and listen to their students so they can plan instruction that builds on these relationships and encourages students to discover and extend important relationships for understanding fraction concepts.

Appendices

APPENDIX A: RESEARCHER AS INSTRUMENT

Spring 2000

During last semester, when I was trying to decide what direction to take my research, fractions became a reoccurring theme.

- While watching a video in one class about the shortage of teachers in the United States, one young man, who was teaching mathematics even though he was certified in another subject area, was interviewed. He explained that he felt comfortable teaching middle school mathematics, except when he had to teach difficult topics such as fractions.
- Later in the semester I interviewed a teacher about her seventh grade mathematics class. She explained that the students had been more successful during first six weeks than the second six weeks. She attributed their difficulties to the fact they were working on fractions during the second six weeks, which was a more difficult topic for students to understand.
- When I presented results on an algebra study at a conference in El Paso, Texas, a participant brought up the difficulties algebra students have with fractions in high school. Another participant carried this on further, adding that community college students also struggle with fractions.

These recent experiences led to me to wonder about what students learn about fractions in elementary school. What do students know and understand about fractions in elementary school that acts as the foundation for future mathematics courses?

My experiences with learning about fractions

Let me start by looking back in my past, before looking towards the future. I remember learning about fractions as a student, but I do not remember many details. I know I was taught rote methods of how to manipulate fractions, and I do not remember my teachers helping me to develop a conceptual understanding of fractions. Since I had so much trouble remembering which number was the numerator and denominator, one of my friends told me to remember that “the denominator is down.” I still use this trick today. I know I learned how to play the school game well. I could look at examples of problems with fractions and figure out the steps to follow. I know that I did not have a strong understanding of fractions. As I child, I thought this was how you were supposed to learn math and did not expect to always understand what I was doing.

While taking my education courses and student teaching, I was exposed to how to teach mathematics so students would develop an understanding of concepts. I realized how I had learned rote procedures as a child. If I had a conceptual understanding of fraction concepts as a child, it was because I made sense of fractions to some extent. Maybe I understood fractions because I used pictorial representations like pizzas or other experiences like measuring when I helped my mom with baking. I think I often developed some understanding of the mathematics I learned in school, but not every student is able to do that. I wanted to learn how to teach differently than I had been taught. I felt like I needed to do more for my students than what had been done for me.

My experiences with teaching fractions

In my first position out of college, I worked as a teacher intern with three fourth grade teachers who were in the process of implementing a new mathematics curriculum. The district had adopted a mathematics textbook for first through third grades, but the curriculum was very new and was only being piloted for fourth grade at the time. The two math coordinators developed a mathematics curriculum that was a compilation of activities from multiple resources. The philosophy of the curriculum matched my own, and I was exposed to a wealth of ideas about how to teach for understanding. I do not remember the specific unit on fractions, but I do remember a conversation about when to teach students to add and subtract fractions. My co-workers were concerned that these topics were introduced to students too soon, while they were still in the process of developing their understanding of fractions. I think I have remembered this idea for so long because it reminds me that I always had to be concerned about what students were ready to learn. If I wanted my students to succeed, I had to start where they were.

When we moved to Austin, Texas, I taught fourth grade at a private Catholic school. Before I started teaching the fraction unit, I laminated pages and pages of construction paper to make fraction circles. Then I went to the local resource center for teachers to use the die cut machine to make fraction circles. Using the die cuts, I cut a variety of fraction pieces to represent whole, halves, thirds, fourths, fifths, sixths, eighths, tenths and twelfths (as I recall), each size from a different color. It took a while to press about thirty-five sets of fractions, but I believed the time and effort would benefit my students. They were going to understand fractions since we would be working with concrete materials, and I would help my students make connections that I had tried to make on my own. I had my students help make sets of fractions using all of these pieces, which we used during the fraction unit. I continued to use these manipulatives while I taught different grade levels and have the materials for when I am teaching again.

When I started the doctorate program at the University of Texas at Austin, I took a mathematics methods course from David Molina. When we learned about fractions, he showed us fraction strips, and we explored how to use this manipulative to develop concepts. I asked the principal at the elementary school where I was teaching if we could purchase some sets of these manipulatives. When she said no, I used blackline masters and made sets of fractions. This time I had the office materials person help with making copies and laminating while parents helped with cutting all of the fraction bars out. Then I started using both the fraction circles and fraction bars for teaching my students.

I felt that these manipulatives helped my students to understand fractions better than they could using the textbook pictures and problems. I also felt like I had grown tremendously in a few years by incorporating manipulatives into the curriculum. As a more experienced teacher, I believe I helped my students learn mathematics and prepare a foundation so they could succeed in future math courses.

I taught fifth grade my last year in the classroom before returning to graduate school full time. Our principal offered the teachers at our school the opportunity to pilot a new mathematics curriculum, *Investigations in Number,* *Data, and Space*. I was the only fifth grade teacher who volunteered to try this curriculum. I was very excited about using these materials because the district mathematics coordinator, Ted Hull, had spoken so favorably about the materials. Several times he told me that he had not seen a better curriculum out there.

Investigations in Number, Data, and Space is a standards-based curriculum developed by TERC with a National Science Foundation (NSF) grant. NSF had funded the development of several curriculums that were based on the National Council of Mathematics Teachers (NCTM) standards published in 1989. The activities in *Investigations* require students not only to think about mathematics, but also to communicate about mathematics. Many of the activities are open-ended and include multiple answers. Other problems had only one solution, but there were many ways to arrive at the answer. Although the curriculum does not focus on the Texas Assessment of Academic Skills (TAAS) or other standardized tests, my students did exceptionally well on the TAAS test. I think they did so well because they developed a belief in their ability to do mathematics and learned that there are many approaches to solving problems. Instead of reading a mathematics problem and thinking, "I have to do division to solve this problem, and I am not very good at division," my students looked at a problem and thought, "There are lots of ways to solve this problem. Let me think about how to solve it."

I taught the fifth-grade unit from *Investigations in Number, Data, and Space* on fractions during December and January. This unit, "Name that Portion,"(Akers, Tierney, Evans, & Murray, 1998) integrates the concepts of fractions, decimals and percents. This was the first time I had seen all of these concepts put together, and I thought it was innovative and useful for students. Since students learn about the relationship between fractions and percents, they could switch between different representations of rational numbers fairly well. Once some of the students realized this, they used this approach with certain problems, such as comparing fractions. It is much easier to determine that 75% is larger than 66 2/3% where as students are frequently confused when comparing 3/4 and 2/3.

My students enjoyed the activities and learned concepts to varying degrees. Some of my students seemed comfortable with all three ways of representing a rational number, where as some students were more comfortable with certain parts or representations and frequently had to refer to the manipulatives from previous activities. One day, another teacher and I had our students work together on some of the games which reinforced decimal concepts from the unit. My students taught Bob's students the games, and they played them during math. This turned out to be a wonderful experience for everyone. My students enjoyed teaching the games and Bob's students enjoyed learning them. I was impressed how well all of the students did with concentrating on the mathematics and working together.

One of my concerns when teaching the unit was about whether all of the students were learning all of the necessary concepts. I knew some of the students really understood the ideas, but I was not sure about everyone. I was also concerned about how well I was tying the concepts together. I felt like I was teaching the unit in a disjointed manner. I was not sure if this was due to the way the unit was written or my own ability to implement it during the first year when I was not comfortable with the curriculum materials.

As part of a course requirement, I did a critique of “Name that Portion.” By reviewing the curriculum without having to implement it in the classroom, I was able to think and write about this unit from a non-teaching perspective. I realized how difficult this unit is for teachers to implement, especially if they are not familiar with or do not agree with the philosophy of the NCTM standards embedded in *Investigations*. This curriculum can provide a strong foundation in fractions for students, but the teacher must be up to the challenge.

This semester I am working with Susan Empson on a research project with teachers who are implementing the third grade fractions unit from *Investigations in Number, Data, and Space* (Tierney, Berle-Carman, Corwin, & Russell, 1998). These teachers are in their first year of implementing this curriculum. My involvement so far has included conducting clinical interviews with students before they started the unit, video taping teachers while teaching specific lessons, sitting in on a teacher interview, giving students a post-test and setting up a database to help with the data analysis. This experience has given me a chance to become familiar with the concepts taught at an earlier grade level and learn how teachers implement the curriculum. In a couple of weeks, I am going to interview teachers about how they implement the curriculum and how they intend to make changes in the unit next year. This experience also helps me think about how I want to conduct my research in the classroom, not only on the dissertation, but in future years as well.

My beliefs

I believe that fractions are a difficult concept for students to learn, but I also think that all students can learn and understand fractions. I think part of this is from the idea that being "mathematically illiterate" is acceptable. People who think it is okay that children are not doing well in mathematics because it is such a difficult subject area are going to not worry about students not understanding fractions because it is considered one of the most difficult topics within mathematics. We have a great deal of evidence that students struggle with fractions, but I believe students can learn if they are taught fractions in a concept based manner.

There are many issues that I believe are integral for teaching mathematics. Students need to develop conceptual understanding of fractions by using a variety of materials and strategies. Students should spend time communicating their strategies and understanding through talking and writing. Students should participate in small group and class discussions, and have opportunities to work independently. All of this requires teachers who have a strong conceptual understanding of fractions. Teachers who do not understand the content thoroughly might be uncomfortable listening to different strategies and trying to make sure students' explanations are correct. Due to the ways we learned about fractions, many adults are more comfortable simply manipulating fractions instead of understanding the underlying concepts. For example, I find myself reverting to the procedure of using the least common denominator to add fractions. To teach with understanding, teachers need to learn new ways to think about fractions. Teachers need to learn this content in the same way that their students need to learn: by actively doing mathematics. I believe professional development should develop teachers' mathematical content knowledge while also assisting them as they incorporate a problem-solving approach to teaching mathematics in their classrooms. Since this is very different from the traditional mathematics approach, teachers need regular support during implementation, including working with colleagues who also use a problem-solving approach.

My values

I believe that all children can learn. All children might not learn the exact same material in the exact same way or in the exact same time, but all children can learn. As an elementary school teacher, I felt that it was my job to prepare students to succeed in school in future classes. Even though I realize that all of my students might not be college-bound, I knew all of my students should master skills and concepts presented in the elementary curriculum. They should be prepared for the middle school mathematics content. One of the challenges I faced was how to present material in such a way as to meet the needs of my different students. I think I was somewhat successful, but I always felt like there was more I should do to meet the individual needs of my students.

I also think mathematics is one of the most important subjects in school, along with reading. In our society, we believe that everyone should read. Literacy is an important part of not only our school system, but of the society in general. It is not acceptable to talk about being illiterate. But our society does not see mathematics in the same way. Often people talk about how they were never any good at mathematics, and it seems socially acceptable. Parents even pass this idea on to their students, and downplay their children's difficulties saying, "I never did well in math either."

I think we need to elevate mathematical literacy to the level of reading literacy. We need to teach everyone about mathematics. Of course this goes beyond just fractions, but fractions have this reputation in our society of being difficult, and it is acceptable if students do not succeed. We need to change this. Fractions are an important building block in understanding mathematics and succeeding in high school and college mathematics courses. Fractions, as well as decimals and the representation between the two, are part of our everyday lives from cooking to following the stock market. Everyone in our community needs to learn how to do mathematics as well as read. If our school-aged children could read, write and understand mathematics, they will be prepared to succeed in future coursework and to go on to college or find jobs when they graduate from high school.

What I expect to see

During the course of my study I expect to find that students are learning about fractions while engaged in a fraction unit. When I watch the students during the class time, I want to see how they learn about fractions. When a child is able to play a game using fractions or answer questions related to fractions, this could be evidence that the child is learning. While the students learn about fractions, I expect to see them make connections between fractions and use these relationships to compare and order fractions. Some of the students will struggle with the concepts at times, but they will grow over time.

Throughout the study, I want to focus on what the child understands. I think I will see "ah-ha!" moments when a student understands how to do a problem. There are going to be several ways that indicate a child understands a concept. For example, when the children play games, I expect that most of them will develop strategies and these strategies will become more sophisticated as the students understand the related concepts better, and many students will be able to explain the strategy orally or in written form. A child who explains how to solve a problem, either to another student or the entire class, shows how he or she understands the problem. A child who understands a concept might draw from previous manipulatives or materials to explain the concept to others.

When I conduct the pre-interviews, I think students will have some strategies for solving a few of the problems based on their informal experiences or learning from previous years in school. I also suspect that some students will have misconceptions about fractions, which might be apparent during the interviews. For example, some students might use whole number relationships to explain that one-fourth is larger than one-third because four is larger than three. I think the students will be willing to try to solve the problems and explain how they solved the problem.

During the clinical interviews, I will look for evidence of learning and evidence of understanding. Students will be able to answer questions in the post-interview that they could not answer in the pre-interview, demonstrating their developing understanding about fractions. If the student can show multiple ways of solving the problem or solve problems that are different than what they did during the unit, it will demonstrate that they understand the concepts. For example, I want to include some problems that are in line with middle school mathematics coursework. If a child understands the underlying concepts, he or she may be able to answer the question or logically explain how to figure out the answer.

I believe most of the students will be able to solve many of the problems after completing the unit. I think the students will use some of the different tools/manipulatives from the unit during the post-interview. I think the students will be willing to try all of the problems and explain their strategies during the post-interview.

What I am willing and not willing to discover

I am willing to discover that students are learning about fractions and have a conceptual understanding of equivalency and relative size of fractions. Some students will be more confident than others, but they will all have some basic understanding of fractional relationships. Some students will struggle more with the concepts than other students, but all of the students will be learning at different paces and in different ways. I am willing to discover that different students prefer and understand some representations better than other representations. I hope to find that students are learning wonderful mathematics in the context of the fraction unit in their class

I am not willing to discover that students have not learned anything during the course of this unit. In a similar manner, I would not want to discover that this approach for teaching fractions works for certain types of students (such as males or Hispanics) but not for other students (such as females or African Americans). Since I will purposefully select a teacher to work with, I plan not to have a situation where the teacher implements a problem-solving approach inconsistent with this philosophy. I hope that I do not see the teacher giving students pages of fraction problems for drills or presenting problems without context or meaning.

Outcomes of my research

The outcomes of my research might be useful for different groups of people. I hope the teacher and classroom experience could be a model of a problem-solving class for other teachers. School administrators or curriculum specialists could share this model with teachers. This would also be useful for pre-service teachers who are learning about reform mathematics and problem-solving approaches to teaching, but are not sure how they will implement it since it is so different from their own classroom experiences.

I hope I can add to the body of research about how students learn fractions. So much of the research in mathematics concentrates on the measurable outcomes after teaching a unit, but the process of how students learn fractions over time is missing. Through this study, I hope to show not only what students learned about fractions, but also the process in which they learned it. Some of the times, I will be able to point to the "ah-ha" moment when the student made a connection. Other mathematics education researchers will be able to use the information from this study as they continue to investigate not only what students learn, but also how students learn fractions.

My interpretivist paradigm

I will conduct this study using an interpretivist paradigm. While conducting the interviews and working with students in the classroom, I will ask them to explain how they solve problems and the strategies they use. Since I am more familiar with fraction concepts, I will be able to relate their approaches to other students from my various experiences and my own understanding. The teacher and I will use our understandings of fraction concepts to interpret how the students learn concepts and the difficulties students face throughout the unit. In analyzing the data, I will explore patterns by individual students and between students and make conjectures about why these patterns exist.

APPENDIX B: PARTICIPANTS

Participants (pseudonyms) sorted by grade and age at the beginning of the study. Students identified with an asterisk (*) participated in mid-interviews.

Name	Grade	Age
Christina*	3	8 years, 3 months
Danielle	3	8 years, 6 months
Joey*	3	8 years, 9 months
Arthur	3	9 years, 1 month
Mark*	4	9 years, 8 months
Allison*	4	9 years, 8 months
Todd	4	10 years, 0 months
Bobby*	4	10 years, 3 months
Krista	4	10 years, 5 months
Elaine	5	10 years, 9 months
Marie*	5	11 years, 0 months
Cheyenne*	5	11 years, 1 month
Elisabeth*	5	11 years, 4 months

APPENDIX C: PARENTAL CONSENT FORM

January 2001

Dear Parent or Guardian:

I am conducting a study on how third, fourth and fifth graders develop models to learn fraction concepts and solve problems while engaged in a fraction unit in their class. I am currently a doctoral candidate at the University of Texas at Austin in mathematics education. This research study will be used to write my dissertation report.

This study consists of observing how students are solving problems in the classroom setting and how children develop models to help solve problems involving fractions. I am conducting this study in your child's third, fourth, and fifth grade class, and I am asking permission to include your child in this study. I plan to work exclusively in your child's class. As many as thirteen students from the class will participate in this study.

While students are engaged in learning from this unit, I will video and/or audio tape class discussions as well as students who are working in small groups or individually. I will ask students to explain how they solved the problems presented by the teacher and make copies of students' solutions. In addition, your child will be interviewed two times outside of class. During these interviews, your child will be asked to solve mathematics problems similar to ones presented in class and explain his or her thinking. These interviews will be videotaped and will last approximately forty-five minutes. Summaries of these interviews will be shared with your child's teacher. The data collected from the interviews and observations will be compiled into a report and individuals will not be identified. The audio and video tapes will be coded so your child's name is not visible and will be kept secured in a file cabinet. I will be transcribing the tapes and they will not be used for any other purpose without your written consent. At the conclusion of this study, the tapes will be kept in a locked filing cabinet for possible future analysis.

Your child's voluntary participation in this study will be confidential, and there are no foreseeable risks or discomforts. Your child's responses will not be linked to his or her name in any written or verbal report of this research project. Possible benefits include having your child learn more about mathematics and helping develop his or her understanding of mathematics concepts.

This report will be submitted as a final project for my dissertation study at the University of Texas at Austin. My advising professor for this study is Susan Empson, Ph.D. and she can be reached at 512-471-3747 or in writing at: The Department of Curriculum and Instruction, University of Texas at Austin, Austin, TX, 78712-1294. A copy of my dissertation will be provided to the school.

Your child's participation in this study is voluntary. Your signature indicates that you have read the information in this letter and have decided to allow your child to participate. You may withdraw your child from this study at any time. Please notify me verbally or in writing if you or your child decides to withdraw from this study. Making a decision not to participate will not affect your or your child's relationship with the University of Texas at Austin or your child's school. If you have questions or would like to read your child's interview summaries or a copy of the report, please contact me via telephone (512-707-1162) or via email at *melaniewenrick@mail.utexas.edu*. Please sign the consent form indicating that your child may participate in this study, and keep this letter for your records.

Sincerely,

Melanie Wenrick

APPENDIX D: PRE-INTERVIEW

Tools available to children: paper and pen (so no erasing), other materials recommended by teacher based on (1) what students used last year and (2) what materials they will have available this year.

Interviewer: Turn on video recorder and give: date, child's full name, classroom and school. Explain to the child:

> **I am interested in finding out about how you solve math problems. I will read a problem out loud, and you can solve the problem any way that makes sense to you. After you have solved the problem, I will ask you questions about how you solved the problem, like Ms. Palmer does. I will read each problem as many times as you need. You can use any tool to solve problems that we have here. You can also solve the problem in your head if you like. Remember, just solve the problem in a way that makes sense to you.**

EQUAL SHARING PROBLEMS

1. **A) Four friends were sharing three cupcakes. They wanted to make sure that everyone had the same amount of cupcake. How much cupcake can each friend have?**

   Probe for how child solved problem.
   Probe for why child made partition that he/she made.

   How would you write one person's share?

   B) [Give only if #1A solved with a viable strategy.] **Can you think of another way the friends could share the cupcakes and get the same amount?**

2. **Who gets more cake: a child who is at a table where 4 children are sharing a cake, or a child who is at a table where 3 children are sharing a cake?**

   If child uses picture to solve, ask if child can solve without picture.

3. [Give only if #1 solved with a viable strategy.] **Three children are sharing five licorice sticks. The children want to make sure that everyone gets the same amount of licorice. How much licorice would one child get?**

Probe for how child solved problem.
Probe for why child made partition that he/she made.
If child leaves a remainder, ask if there is a way to share the remaining *licorice between the 3 children.*

How would you write one person's share?

4. A) [Give only if #3 is solved with a viable strategy.] **Six people are sharing ten pancakes. How much would one person receive if they share the pancakes so everyone has the same amount?**
Probe for how child solved problem.
Probe for why child made partition that he/she made.

How would you write one person's share?

B) *Show work from problem #3 to student again.* **Someone told me that this problem was a lot like the licorice problem. Why do you think they would say that?**

5. A) You are making banana nut bread. The recipe says you need 4 bananas to make 6 loaves. How many bananas do you need to make 12 loaves?
Probe for how child solved problem.

B) How many bananas do you need if you only want to make 3 loaves of bread?

6. **A) The children in Ms. Jones' class were solving an equal sharing problem, where all the cakes were the same size. Some got one-half of a cake for their answer. Others got two-fourths of a cake for their answer. Are these the same amounts of cake or different amounts of cake?**

   B) How do you know?
   Probe for how child made decision.

   C) [Give only if child answers part 6A correctly and gives a viable explanation for part 6B.] **One student said the answer could also be three-sixths of a cake. Is this student correct?**
   [If child says student is correct.] **Why?**
   [If child says student is not correct.] **Why not?**

   D) [Give only if child gives a viable explanation for part 6C.] **Can you think of another answer that would be correct?**
   Probe for child's reasoning about why this additional answer is correct.

7. **A)** [Give only if student gave viable answers for #6 A, B, and C.] **Another day the children in Ms. Jones' class were solving a different equal sharing problem, where all the cakes were the same size. This time some got two-thirds of a cake for their answer. Others got six-ninths of a cake for their answer. Are these the same amounts of cake or different amounts of cake?**

   B) How do you know?
   Probe for how child made decision.

   C) [Give only if child answers part 7A correctly and gives a viable explanation for part 7B.] **One student said the answer could also be eight-twelfths of a cake. Is this student correct?**
   [If child says student is correct.] **Why?**
   [If child says student is not correct.] **Why not?**

   D) [Give only if child gives a viable explanation for part 7C.] **Can you think of another answer that would be correct?**
   Probe for child's reasoning about why this additional answer is correct.

PAPER FOLDING PROBLEMS

8. Fold paper in half. **When I open up this paper, how many pieces will I have made?**

How big is this piece (point to the half page on top) **compared to the whole piece of paper?** Unfold the paper. Make sure student sees the two pieces and that one piece is one-half of a whole.

9. This next time, we are going to fold the paper in half three times. If we fold the paper in half three times, how many pieces do you think we will make?

Why do you think it will make ____ pieces?

Have the student fold the paper in half three times. Ask student to stop when they have finished folding the paper.

(Point to the paper still folded.) **How big is this one piece compared to the whole piece of paper? Why?**

How big will two pieces be compared to the whole? How do you know?

Have students unfold the paper now. Have students count the number of pieces.

If students predicted 8, ask, **How did you know it would have 8 pieces before you did the folding?**

If students predicted something other than 8, ask, **Why do you think your prediction was different than what you found when you folded and counted?**

(Point to one of the sections.) **How big is this one piece compared to the whole piece of paper? Why?**

How big will two pieces be compared to the whole? How do you know?

10. **This next time, we are going to fold the paper in half and then we are going to fold it into thirds. If we fold the paper in half and then into thirds, how many pieces do you think we will make?**

Why do you think it will make ____ pieces?

Have the student fold the paper in half and then in thirds. Ask student to stop when they have finished folding the paper.

(Point to the paper still folded.) **How big is this one piece compared to the whole piece of paper? Why?**

How big will two pieces be compared to the whole? How do you know?

Have students unfold the paper now. Have students count the number of pieces.

If students predicted 6, ask, **How did you know it would have 6 pieces before you did the folding?**

If students predicted something other than 6, ask, **Why do you think your prediction was different than what you found when you folded and counted?**

(Point to one of the sections.) **How big is this one piece compared to the whole piece of paper? Why?**

How big will two pieces be compared to the whole? How do you know?

11. [Give only if students were able to do both 10 and 11 at least when they had the papers open completely.] Take the folded pieces of paper from the two previous questions. **We folded this paper into half three times. We folded this paper into half and then into thirds. If we just think about one piece from each paper, which one has the smaller piece? How do you know?**

SYMBOLIC PROBLEMS

12. Arrange these fractions in order from smallest to largest: One-fourth, one-fifth, and one-third? How do you know ___ is the smallest? How do you know ___ is the largest?

Probe for how child determines the order of the fractions.
Probe for how child decided which fraction is the smallest.
Probe for how child decided which fraction is the largest.

13. Are these fractions different amounts or the same amounts: Two-thirds and three-fourths. How do you know?

Probe for how child decided whether fractions are the same or different amounts.

If student says different amounts ask, **Which one is larger? How do you know?**

14. Are these fractions different amounts or the same amount: Two-twelfths and one-sixth? How do you know?

Probe for how child decided whether fractions are the same or different amounts.

CHALLENGE PROBLEMS

(Only given if extra time and student has answered most of the other questions without much difficulty.)

15. I bought two and five-eighths yards of material to make an outfit. If I only use one and three-fourth yards of the material to make a dress, how much material will I have left over?

Probe for how child solved problem.

16. A chocolate cake recipe uses two-thirds cup of butter for the batter and one-half cup of butter for the icing. How much butter does this recipe require in all?

Probe for how child solved problem.

17. There were eighteen children at a school pizza party. If the teacher wants to allow each child one-third of a pizza, how many pizzas should the teacher order?

Probe for how child solved problem.

18. Mom makes small apple tarts, using three-quarters of an apple for each small tart. She has twenty apples. How many small tarts can she make?

Probe for how child solved problem.

19. It takes three-fourths of an hour to paint one-third of a room. How long will it take to paint the whole room?

Probe for how child solved problem.

20. Kayla has a miniature doll house mirror that is one-half inch wide and one-third inch tall. What is the area of the mirror in square inches?

Probe for how child solved problem.

ENDING QUESTION – IF NECESSARY

21. [Give only if need a problem to end on a positive note.] **Stacy has twelve pencils. She wants to give the pencils to three friends. If she gives each friend the same number of pencils, how many pencils will each friend receive?**

APPENDIX E: MID-INTERVIEW

Tools available to children: paper and pen (so no erasing), two-colored counters, linking cubes, ruler, fractions strips (made by students).

Interviewer: Turn on video recorder and give: date, child's full name, classroom and school. Explain to the child:

> **I am interested in finding out about how you solve math problems. I will read a problem out loud, and you can solve the problem any way that makes sense to you. After you have solved the problem, I will ask you questions about how you solved the problem, like Ms. Palmer does. I will read each problem as many times as you need. You can use any tool to solve problems that we have here. You can also solve the problem in your head if you like. Remember, just solve the problem in a way that makes sense to you.**

EQUAL SHARING PROBLEMS

1. A) Twelve friends were sharing thirty-two pancakes. They wanted to make sure that everyone had the same amount of pancake. How much pancake can each friend have?

Probe for how child solved problem.
Probe for why child made partition that he/she made.

How would you write one person's share?

If child writes 1, 1/2, 1/6 (or 2/12), ask if the child can combine the fractions 1/2 and 1/6 (or 2/12).

B) [Give only if #1A solved with a viable strategy.] **Is there another answer for this problem that would mean the same amount?**

2. **A) Four children are sharing six licorice sticks. The children want to make sure that everyone gets the same amount of licorice. How much licorice would one child get?**
   Probe for how child solved problem.
   Probe for why child made partition that he/she made.
   If child leaves a remainder, ask if there is a way to share the remaining *licorice between the 4 children.*

   How would you write one person's share?

   B) [Give only if #2A solved with a viable strategy.] **Is there another answer for this problem that would mean the same amount?**

3. **A)** [Give only if #1 or 2 is solved with a viable strategy.] **Twelve people are sharing eighteen candy bars. How much would one person receive if they share the candy bars so everyone has the same amount?**
   Probe for how child solved problem.
   Probe for why child made partition that he/she made.

   How would you write one person's share?

   B) [Give only if #3A solved with a viable strategy.] **Is there another answer for this problem that would mean the same amount?**

   C) *Show work from problem #2 to student again.* **Someone told me that this problem was a lot like the licorice problem. Why do you think they would say that?**

4. **Ask student about the representations and/or manipulatives used for the first three questions.**

   If the student drew different shapes: **Why did you use a circle (rectangle, line) for this problem, but not for the other problems?**

   If the student drew the same shape: **Why did you use a circle (rectangle, line) for all of these problems?**

   If the student used the same manipulative: **Why did you use cubes (counters) to solve all of the problems?**

   If the student used different manipulatives: **Why did you use cubes (counters) to solve this problem and use counters (cubes) to solve this other one?**

   If the students used a combination of manipulatives and drawings: **Why did you use cubes (counters) to solve this problem, but drew a picture to solve this other problem?**

5. **A) The children in Ms. Lee's class were solving an equal sharing problem, where all the cakes were the same size. Some got two-eighths of a cake for their answer. Others got three-twelfths of a cake for their answer. Are these the same amounts of cake or different amounts of cake?**

   B) How do you know?
   Probe for how child made decision.

   C) [Give only if child gives a viable explanation for part 5A and 5B.] **What's another way to say that amount of cake with fractions?**
   Probe for child's reasoning about why this additional answer is correct.

6. **A)** [Give only if student gave viable answers for #5 A and B.] **Another day the children in Ms. Lee's class were solving a different equal sharing problem, where all the cakes were the same size. Some students said the answer was ten-twelfths of a cake. Others said five-sixths of a cake for their answer. Are these the same amounts of cake or different amounts of cake?**

   B) How do you know?
   Probe for how child made decision.

   C) [Give only if child gives a viable explanation for part A and B.] **What's another way to say that amount of cake with fractions?**
   Probe for child's reasoning about why this additional answer is correct.

7. **Ask student about the representations and/or manipulatives used for the fifth and sixth questions.**

   If the student drew different shapes: **Why did you use a circle (rectangle, line) for this problem, but not for the other problem?**

   If the student drew the same shape: **Why did you use a circle (rectangle, line) for both of these problems?**

   If the student used the same manipulative: **Why did you use cubes (counters) to solve both of these problems?**

   If the student used different manipulatives: **Why did you use cubes (counters) to solve this problem and use counters (cubes) to solve this other one?**

   If the students used a combination of manipulatives and drawings: **Why did you use cubes (counters) to solve this problem, but drew a picture to solve this other problem?**

SYMBOLIC PROBLEMS

8. **Arrange these fractions in order from smallest to largest: Four-sixths, four-twelfths, and four-eighths. How do you know ___ is the smallest? How do you know ___ is the largest?**
    Probe for how child determines the order of the fractions.
    Probe for how child decided which fraction is the smallest.
    Probe for how child decided which fraction is the largest.

9. **Are these fractions different amounts or the same amounts: Three-fourths and four-thirds. How do you know?**
    Probe for how child decided whether fractions are the same or different amounts.

    If student says different amounts ask, **Which one is larger? How do you know?**

10. **Are these fractions different amounts or the same amounts: Two-thirds and four-sixths. How do you know?**
    Probe for how child decided whether fractions are the same or different amounts.

11. **Are these fractions different amounts or the same amount: three-sixteenths and five-eighths? How do you know?**
    Probe for how child decided whether fractions are the same or different amounts.

    If student says different amounts ask, **Which one is larger? How do you know?**

    If child uses fraction strips to solve this problem ask, **Is there another way that you could figure out which one is larger without using the fraction strips?**

APPENDIX F: POST-INTERVIEW

Tools available to children: paper and pen (so no erasing), two-colored counters, linking cubes, colored-tiles, pattern blocks, base ten blocks, ruler, fractions strips (made by students).

Interviewer: Turn on video recorder and give: date, child's full name, classroom and school. Explain to the child:

> **I am interested in finding out about how you solve math problems. I will read a problem out loud, and you can solve the problem any way that makes sense to you. After you have solved the problem, I will ask you questions about how you solved the problem, like Ms. Palmer does. I will read each problem as many times as you need. You can use any tool to solve problems that we have here. You can also solve the problem in your head if you like. Remember, just solve the problem in a way that makes sense to you.**

EQUAL SHARING PROBLEMS

1. A) Twelve friends were sharing forty-six candy bars. They wanted to make sure that everyone had the same amount of candy bar. How much candy bar can each friend have?

Probe for how child solved problem.
Probe for why child made partition that he/she made.

How would you write one person's share?

If child writes 1, 1/2, 1/3 (or2/6 or 4/12), ask if the child can combine the fractions 1/2 and 1/3 (or 2/6 or 4/12).

B) [Give only if #1A solved with a viable strategy.] **Is there another answer for this problem that would mean the same amount?**

Probe for how student determines/finds equivalent fractions.

2. A) Eight teachers are sharing six apples. How much would one teacher receive if they share the apples so everyone has the same amount?

Probe for how child solved problem.
Probe for why child made partition that he/she made.

How would you write one teacher's share?

B) [Give only if #2A solved with a viable strategy.] **Is there another answer for this problem that would mean the same amount?**

Probe for how student determines/finds equivalent fractions.

3. [Give only if #2A solved with a viable strategy.] *Keep the child's solution from question number 3 on the table as a reference for this problem.* **Using the question and your answer from before where eight teachers were sharing six apples, can you solve the following problems…?**

A) **Sixteen teachers are sharing twelve apples, how much would each teacher receive?**

Probe for how child solved problem using their answer to question 2.

B) **Eight teachers are sharing twelve apples, how much would one teacher receive?**

Probe for how child solved problem using their answer to question 2.

C) [Give only if #3A and #3B were solved with a viable strategy.] **Twelve teachers are sharing nine apples, how much would one teacher receive?**

Probe for how child solved problem using their answer to question 2.

4. **A) The children in Ms. Lee's class were solving an equal sharing problem, where all the pancakes were the same size. Some got two-fifths of a pancake for their answer. Others got four-tenths of a pancake for their answer. Are these the same amounts of pancake or different amounts of pancake?**

   B) How do you know?

   Probe for how child made decision.
   If student says that they are multiplying or dividing by 2/2, ask: **Why does that work?**
   If student says that multiplying by 2/2 is the same as multiplying or dividing by 1 ask: **How could you explain or show why multiplying or dividing by 2/2 works to a classmate?**

   C) [Give only if child gives a viable explanation for part A and B.] **What's another way to say that amount of pancake with fractions?**
   Probe for child's reasoning about why this additional answer is correct.

   D) [Give some of these additional amounts only if child gives a viable answer(s) for part C that is using doubling strategies.] **Could six-fifteenths (or 10/25, 14/35) be the same amount as two-fifth and four-tenths?**
   Probe for child's reasoning about why this answer is also equal/not equal.

CHALLENGE/EXTENSION PROBLEMS

5. **A cookie recipe calls for 4 cups of flour and makes 32 big cookies. Claire wants to make just 24 big cookies. How many cups of flour does she need?**
   Probe for how child determines the ratio between cups of flour and cookies.
   Probe for how child uses the ratio relationship to determine the amount of flour needed for 24 cookies.

6. **Aldo wants to make 4 batches of brownies. Each batch requires 3 3/8 cups of flour, 2 1/4 cups of sugar, and 1 1/2 teaspoons of vanilla. How many cups of flour will be needed?**
   Probe for how child solved problem.

7. **A) A developer has 31 1/2 acres of land that he plans to sell in 3/4 acre lots. How many lots can he make from this land?**
   Probe for how child solved problem.

   B) [Give only if child is unable to figure out #7A, but they have a viable strategy.] **What if the developer only has 5 1/4 acres of land that he plans to sell in 3/4 acre lots? How many lots can he make from this land?**
   Probe for how child solved problem.

8. [Only give this problem if #7 solved correctly and it seems to be an easy problem for student.] **A developer has 43 1/3 acres of land that he plans to sell in 1 2/3 acre lots. How many lots can he make from this land?**
   Probe for how child solved problem.

9. **3 1/3 – 1 5/9 =**
   Probe for how child solved problem.
   Does the student recognize and use the relationship between thirds and ninths?
   Does the student realize that borrowing with fractions is different than with whole numbers?
   If the child ends up with more than one fraction ask, **Can you combine those fractions into a fraction with one denominator?**

10. **A) 5 8/10 + 2 3/5 =**
   Probe for how child solved problem.
   Does the student recognize and use the relationship between tenths and fifths?

   B) [Give only if #10A solved with a viable strategy.] **Is there another answer for this problem that would mean the same amount?**
   Probe for how student determines/finds equivalent fractions.

SYMBOLIC PROBLEMS

11. A) *While reading this question, give students index cards with one fraction written on each one.* **Arrange these fractions in order from smallest to largest: two-thirds, three-fourths, one-sixth, and one-half? How do you know ___ is the smallest? How do you know ___ is the largest?**

Probe for how child determines the order of the fractions.

Probe for how child decided which fraction is the smallest.

Probe for how child decided which fraction is the largest.

B) If a student converts the numbers to a common denominator to compare the amounts ask, **Is there a way you can check your answer without converting all of these fractions to twelfths (the common denominator)?**

12. *A) While reading this question, give students index cards with one fraction written on each one.* **Arrange these fractions in order from smallest to largest: seven-fifteenths, two-sevenths, and six-tenths? How do you know ___ is the smallest? How do you know ___ is the largest?**

Probe for how child determines the order of the fractions.

Probe for how child decided which fraction is the smallest.

Probe for how child decided which fraction is the largest.

B) *If this problem is too difficult for a specific student, ask the student to* ***compare two of the fractions at a time such as: 6/10 and 2/7, 2/7 and 7/15, 6/10 and 7/15****.*

13. Are these fractions different amounts or the same amounts: four-sixths and three-fifths. How do you know?

Probe for how child decided whether fractions are the same or different amounts.

If student says different amounts ask, **Which one is larger? How do you know?**

If the child gives an incorrect answer because they are only comparing numerators, denominators or the difference between ask, **Can you think of another way to prove that these fractions are different amounts?**

14. Are these fractions different amounts or the same amounts: two-eighths and three-twelfths. How do you know?

Probe for how child decided whether fractions are the same or different amounts.

If the child uses the fraction kit to solve the problem ask, **Can you think of another way to prove that these fractions are the same amount without using the fraction kit?**

APPENDIX G: FREQUENCY OF PERSPECTIVES

Percentage of codes for each perspective by interview. Ordered by descending overall frequency of occurrence.

Perspectives	Interview			Overall
	Pre	Mid	Post	
Part-Whole	38.2%	33.3%	17.5%	28.3%
Equivalence	11.2%	24.2%	14.9%	16.0%
Within-Fraction	6.7%	9.1%	24.6%	14.9%
Unit Fraction	10.1%	12.1%	17.5%	13.8%
Pieces	2.2%	10.6%	10.5%	7.8%
Limited	20.2%	0.0%	1.8%	7.4%
Between-Fraction	6.7%	7.6%	7.0%	7.1%
Transform	4.5%	3.0%	6.1%	4.8%

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